UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 8th June, 2006 9 to 12

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

(i) Give an example of an integral domain that is not a unique factorization domain.

(ii) For which integers n is $\mathbb{Z}/n\mathbb{Z}$ an integral domain?

2H Geometry

Show that the Gaussian curvature K at an arbitrary point (x, y, z) of the hyperboloid $z = xy$, as an embedded surface in $\mathbf{R}^3$, is given by the formula

$$K = -1/(1 + x^2 + y^2)^2.$$

3F Analysis II

Define what it means for a function $f : \mathbf{R}^2 \to \mathbf{R}$ to be *differentiable* at a point (a, b). If the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are defined and continuous on a neighbourhood of (a, b), show that f is differentiable at (a, b).

4F Metric and Topological Spaces

Which of the following are topological spaces? Justify your answers.

(i) The set $X = \mathbf{Z}$ of the integers, with a subset A of X called "open" when A is either finite or the whole set X;

(ii) The set $X = \mathbf{Z}$ of the integers, with a subset A of X called "open" when, for each element $x \in A$ and every even integer n, $x + n$ is also in A.

5D Complex Methods

The transformation

$$w = i\left(\frac{1-z}{1+z}\right)$$

maps conformally the interior of the unit disc D onto the upper half-plane H_+, and maps the upper and lower unit semicircles C_+ and C_- onto the positive and negative real axis $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

Consider the Dirichlet problem in the upper half-plane:

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0 \quad \text{in} \quad H_+\,; \qquad f(u,v) = \begin{cases} 1 & \text{on } \mathbb{R}_+, \\ 0 & \text{on } \mathbb{R}_-. \end{cases}$$

Its solution is given by the formula

$$f(u,v) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{u}{v}\right).$$

Using this result, determine the solution to the Dirichlet problem in the unit disc:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \quad \text{in} \quad D; \qquad F(x,y) = \begin{cases} 1 & \text{on } C_+, \\ 0 & \text{on } C_-. \end{cases}$$

Briefly explain your answer.

6A Methods

If T_{ij} is a second rank tensor such that $b_i T_{ij} c_j = 0$ for every vector $\mathbf{b}$ and every vector $\mathbf{c}$, show that $T_{ij} = 0$.

Let S be a closed surface with outward normal $\mathbf{n}$ that encloses a three-dimensional region having volume V. The position vector is $\mathbf{x}$. Use the divergence theorem to find

$$\int_S (\mathbf{b}\cdot\mathbf{x})(\mathbf{c}\cdot\mathbf{n})\,dS$$

for constant vectors $\mathbf{b}$ and $\mathbf{c}$. Hence find

$$\int_S x_i n_j \, dS,$$

and deduce the values of

$$\int_S \mathbf{x}\cdot\mathbf{n}\,dS \quad \text{and} \quad \int_S \mathbf{x}\times\mathbf{n}\,dS.$$

7B Quantum Mechanics

Define the quantum mechanical operators for the angular momentum $\hat{\mathbf{L}}$ and the total angular momentum $\hat{L}^2$ in terms of the operators $\hat{\mathbf{x}}$ and $\boldsymbol{\nabla}$. Calculate the commutators $[\hat{L}_i, \hat{L}_j]$ and $[\hat{L}^2, \hat{L}_i]$.

8C Statistics

One hundred children were asked whether they preferred crisps, fruit or chocolate. Of the boys, 12 stated a preference for crisps, 11 for fruit, and 17 for chocolate. Of the girls, 13 stated a preference for crisps, 14 for fruit, and 33 for chocolate. Answer each of the following questions by carrying out an appropriate statistical test.

(a) Are the data consistent with the hypothesis that girls find all three types of snack equally attractive?

(b) Are the data consistent with the hypothesis that boys and girls show the same distribution of preferences?

9C Markov Chains

A hungry student always chooses one of three places to get his lunch, basing his choice for one day on his gastronomic experience the day before. He sometimes tries a sandwich from Natasha's Patisserie: with probability 1/2 this is delicious so he returns the next day; if the sandwich is less than delicious, he chooses with equal probability 1/4 either to eat in Hall or to cook for himself. Food in Hall leaves no strong impression, so he chooses the next day each of the options with equal probability 1/3. However, since he is a hopeless cook, he never tries his own cooking two days running, always preferring to buy a sandwich the next day. On the first day of term the student has lunch in Hall. What is the probability that 60 days later he is again having lunch in Hall?

[*Note* $0^0 = 1$.]

SECTION II

10H Linear Algebra

(a) Define what is meant by the *trace* of a complex $n \times n$ matrix A. If T denotes an $n \times n$ invertible matrix, show that A and TAT^{-1} have the same trace.

(b) If $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, show that the endomorphism of $\mathbf{C}^r$ defined by the matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \ldots & \lambda_1^r \\ \vdots & \ldots & \vdots \\ \lambda_r & \ldots & \lambda_r^r \end{pmatrix}$$

has trivial kernel, and hence that the same is true for the transposed matrix Λ^t.

For arbitrary complex numbers $\lambda_1, \ldots, \lambda_n$, show that the vector $(1, \ldots, 1)^t$ is not in the kernel of the endomorphism of $\mathbf{C}^n$ defined by the matrix

$$\begin{pmatrix} \lambda_1 & \ldots & \lambda_n \\ \vdots & \ldots & \vdots \\ \lambda_1^n & \ldots & \lambda_n^n \end{pmatrix},$$

unless all the λ_i are zero.

[*Hint: reduce to the case when $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, with $r \leqslant n$, and each λ_j for $j > r$ is either zero or equal to some λ_i with $i \leqslant r$. If the kernel of the endomorphism contains $(1, \ldots, 1)^t$, show that it also contains a vector of the form $(m_1, \ldots, m_r, 0, \ldots, 0)^t$ with the m_i strictly positive integers.*]

(c) Assuming the fact that any complex $n \times n$ matrix is conjugate to an upper-triangular one, prove that if A is an $n \times n$ matrix such that A^k has zero trace for all $1 \leqslant k \leqslant n$, then $A^n = 0$.

11E Groups, Rings and Modules

Suppose that R is a ring. Prove that $R[X]$ is Noetherian if and only if R is Noetherian.

12H Geometry

Describe the stereographic projection map from the sphere S^2 to the extended complex plane $\mathbf{C}_\infty$, positioned equatorially. Prove that $w, z \in \mathbf{C}_\infty$ correspond to antipodal points on S^2 if and only if $w = -1/\bar{z}$. State, without proof, a result which relates the rotations of S^2 to a certain group of Möbius transformations on $\mathbf{C}_\infty$.

Show that any circle in the complex plane corresponds, under stereographic projection, to a circle on S^2. Let C denote any circle in the complex plane other than the unit circle; show that C corresponds to a great circle on S^2 if and only if its intersection with the unit circle consists of two points, one of which is the negative of the other.

[*You may assume the result that a Möbius transformation on the complex plane sends circles and straight lines to circles and straight lines.*]

13F Analysis II

State precisely the inverse function theorem for a smooth map F from an open subset of $\mathbf{R}^2$ to $\mathbf{R}^2$.

Define $F : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$F(x, y) = (x^3 - x - y^2, y).$$

Determine the open subset of $\mathbf{R}^2$ on which F is locally invertible.

Let C be the curve $\{(x, y) \in \mathbf{R}^2 : x^3 - x - y^2 = 0\}$. Show that C is the union of the two subsets $C_1 = \{(x, y) \in C : x \in [-1, 0]\}$ and $C_2 = \{(x, y) \in C : x \geqslant 1\}$. Show that for each $y \in \mathbf{R}$ there is a unique $x = p(y)$ such that $(x, y) \in C_2$. Show that F is locally invertible at all points of C_2, and deduce that $p(y)$ is a smooth function of y.

[*A function is said to be smooth when it is infinitely differentiable.*]

14H Complex Analysis

Assuming the principle of the argument, prove that any polynomial of degree n has precisely n zeros in $\mathbf{C}$, counted with multiplicity.

Consider a polynomial $p(z) = z^4 + az^3 + bz^2 + cz + d$, and let R be a positive real number such that $|a|R^3 + |b|R^2 + |c|R + |d| < R^4$. Define a curve $\Gamma : [0,1] \to \mathbf{C}$ by

$$\Gamma(t) = \begin{cases} p(Re^{\pi i t}) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}\ , \\ (2-2t)p(iR) + (2t-1)p(R) & \text{for } \frac{1}{2} \leqslant t \leqslant 1\ . \end{cases}$$

Show that the winding number $n(\Gamma, 0) = 1$.

Suppose now that $p(z)$ has real coefficients, that $z^4 - bz^2 + d$ has no real zeros, and that the real zeros of $p(z)$ are all strictly negative. Show that precisely one of the zeros of $p(z)$ lies in the quadrant $\{x + iy : x > 0, y > 0\}$.

[*Standard results about winding numbers may be quoted without proof; in particular, you may wish to use the fact that if* $\gamma_i : [0,1] \to \mathbf{C}$, $i = 1, 2$, *are two closed curves with* $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t)|$ *for all* t, *then* $n(\gamma_1, 0) = n(\gamma_2, 0)$.]

15G Methods

(a) Find the Fourier sine series of the function

$$f(x) = x$$

for $0 \leqslant x \leqslant 1$.

(b) The differential operator L acting on y is given by

$$L[y] = y'' + y'.$$

Show that the eigenvalues λ in the eigenvalue problem

$$L[y] = \lambda y, \quad y(0) = y(1) = 0,$$

are given by $\lambda = -n^2\pi^2 - \frac{1}{4}$, $\quad n = 1, 2, \ldots$, and find the corresponding eigenfunctions $y_n(x)$.

By expressing the equation $L[y] = \lambda y$ in Sturm-Liouville form or otherwise, write down the orthogonality relation for the y_n. Assuming the completeness of the eigenfunctions and using the result of part (a), find, in the form of a series, a function $y(x)$ which satisfies

$$L[y] = xe^{-x/2}$$

and $y(0) = y(1) = 0$.

16B Quantum Mechanics

The expression $\Delta_\psi A$ denotes the uncertainty of a quantum mechanical observable A in a state with normalised wavefunction ψ. Prove that the Heisenberg uncertainty principle

$$(\Delta_\psi x)(\Delta_\psi p) \geqslant \frac{\hbar}{2}$$

holds for all normalised wavefunctions $\psi(x)$ of one spatial dimension.

[*You may quote Schwarz's inequality without proof.*]

A Gaussian wavepacket evolves so that at time t its wavefunction is

$$\psi(x,t) = (2\pi)^{-\frac{1}{4}} \Big(1 + i\hbar t\Big)^{-\frac{1}{2}} \exp\Big(-\frac{x^2}{4(1+i\hbar t)}\Big)\,.$$

Calculate the uncertainties $\Delta_\psi x$ and $\Delta_\psi p$ at each time t, and hence verify explicitly that the uncertainty principle holds at each time t.

$\Big[$*You may quote without proof the results that if* $\mathrm{Re}(a) > 0$ *then*

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a^*}\right) x^2 \exp\left(-\frac{x^2}{a}\right) dx = \frac{1}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{|a|^3}{(\mathrm{Re}(a))^{\frac{3}{2}}}$$

and

$$\int_{-\infty}^{\infty} \left(\frac{d}{dx}\exp\left(-\frac{x^2}{a^*}\right)\right)\left(\frac{d}{dx}\exp\left(-\frac{x^2}{a}\right)\right) dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{|a|}{(\mathrm{Re}(a))^{\frac{3}{2}}}\,.\Big]$$

17G Electromagnetism

Write down Maxwell's equations in vacuo and show that they admit plane wave solutions in which

$$\mathbf{E}(\mathbf{x},t) = Re\left(\mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\right), \quad \mathbf{k}\cdot\mathbf{E}_0 = 0,$$

where $\mathbf{E}_0$ and $\mathbf{k}$ are constant vectors. Find the corresponding magnetic field $\mathbf{B}(\mathbf{x},t)$ and the relationship between ω and $\mathbf{k}$.

Write down the relations giving the discontinuities (if any) in the normal and tangential components of $\mathbf{E}$ and $\mathbf{B}$ across a surface $z = 0$ which carries surface charge density σ and surface current density $\mathbf{j}$.

Suppose that a perfect conductor occupies the region $z < 0$, and that a plane wave with $\mathbf{k} = (0,0,-k)$, $\mathbf{E}_0 = (E_0, 0, 0)$ is incident from the vacuum region $z > 0$. Show that the boundary conditions at $z = 0$ can be satisfied if a suitable reflected wave is present, and find the induced surface charge and surface current densities.

18A Fluid Dynamics

State and prove Bernoulli's theorem for a time-dependent irrotational flow of an inviscid fluid.

A large vessel is part-filled with inviscid liquid of density ρ. The pressure in the air above the liquid is maintained at the constant value $P + p_a$, where p_a is atmospheric pressure and $P > 0$. Liquid can flow out of the vessel along a cylindrical tube of length L. The radius a of the tube is much smaller than both L and the linear dimensions of the vessel. Initially the tube is sealed and is full of liquid. At time $t = 0$ the tube is opened and the liquid starts to flow. Assuming that the tube remains full of liquid, that the pressure at the open end of the tube is atmospheric and that P is so large that gravity is negligible, determine the flux of liquid along the tube at time t.

19D Numerical Analysis

(a) Define the QR factorization of a rectangular matrix and explain how it can be used to solve the least squares problem of finding an $x^* \in \mathbb{R}^n$ such that

$$\|Ax^* - b\| = \min_{x \in \mathbb{R}^n} \|Ax - b\|, \quad \text{where} \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad m \geqslant n,$$

and the norm is the Euclidean distance $\|y\| = \sqrt{\sum_{i=1}^{m} |y_i|^2}$.

(b) Define a Householder transformation (reflection) H and prove that H is an orthogonal matrix.

(c) Using Householder reflection, solve the least squares problem for the case

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix},$$

and find the value of $\|Ax^* - b\| = \min_{x \in \mathbb{R}^2} \|Ax - b\|$.

20C Optimization

Explain what is meant by a two-person zero-sum game with payoff matrix $A = (a_{ij} : 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n)$ and what is meant by an optimal strategy $p = (p_i : 1 \leqslant i \leqslant m)$.

Consider the following betting game between two players: each player bets an amount $1, 2, 3$ or 4; if both bets are the same, then the game is void; a bet of 1 beats a bet of 4 but otherwise the larger bet wins; the winning player collects both bets. Write down the payoff matrix A and explain why the optimal strategy $p = (p_1, p_2, p_3, p_4)^T$ must satisfy $(Ap)_i \leqslant 0$ for all i. Hence find p.

END OF PAPER