UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 7th June, 2006 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

State Sylvester's law of inertia.

Find the rank and signature of the quadratic form q on $\mathbf{R}^n$ given by

$$q\left(x_1, \ldots, x_n\right)=\left(\sum_{i=1}^n x_i\right)^2-\sum_{i=1}^n x_i^2 .$$

2E Groups, Rings and Modules

(i) Give the definition of a Euclidean domain and, with justification, an example of a Euclidean domain that is not a field.

(ii) State the structure theorem for finitely generated modules over a Euclidean domain.

(iii) In terms of your answer to (ii), describe the structure of the $\mathbb{Z}$-module M with generators $\{m_1, m_2, m_3\}$ and relations $2m_3 = 2m_2$, $4m_2 = 0$.

3F Analysis II

Define uniform convergence for a sequence $f_1, f_2, \ldots$ of real-valued functions on an interval in $\mathbf{R}$. If (f_n) is a sequence of continuous functions converging uniformly to a (necessarily continuous) function f on a closed interval $[a, b]$, show that

$$\int_a^b f_n(x)\, dx \rightarrow \int_a^b f(x)\, dx$$

as $n \rightarrow \infty$.

Which of the following sequences of functions $f_1, f_2, \ldots$ converges uniformly on the open interval $(0,1)$? Justify your answers.

(i) $f_n(x)=1/(nx)$;

(ii) $f_n(x)=e^{-x/n}$.

4F Metric and Topological Spaces

Which of the following subspaces of Euclidean space are connected? Justify your answers.

(i) $\{(x,y,z) \in \mathbf{R}^3 : z^2 - x^2 - y^2 = 1\}$;

(ii) $\{(x,y) \in \mathbf{R}^2 : x^2 = y^2\}$;

(iii) $\{(x,y,z) \in \mathbf{R}^3 : z = x^2 + y^2\}$.

5A Methods

Describe briefly the method of Lagrange multipliers for finding the stationary values of a function $f(x,y)$ subject to a constraint $g(x,y) = 0$.

Use the method to find the smallest possible surface area (including both ends) of a circular cylinder that has volume V.

6G Electromagnetism

Given that the electric field $\mathbf{E}$ and the current density $\mathbf{j}$ within a conducting medium of uniform conductivity σ are related by $\boldsymbol{j} = \sigma\mathbf{E}$, use Maxwell's equations to show that the charge density ρ in the medium obeys the equation

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\epsilon_0}\rho.$$

An infinitely long conducting cylinder of uniform conductivity σ is set up with a uniform electric charge density ρ_0 throughout its interior. The region outside the cylinder is a vacuum. Obtain ρ within the cylinder at subsequent times and hence obtain $\mathbf{E}$ and $\mathbf{j}$ within the cylinder as functions of time and radius. Calculate the value of $\mathbf{E}$ outside the cylinder.

7B Special Relativity

A_1 moves at speed v_1 in the x-direction with respect to A_0. A_2 moves at speed v_2 in the x-direction with respect to A_1. By applying a Lorentz transformation between the rest frames of A_0, A_1, and A_2, calculate the speed at which A_0 observes A_2 to travel.

A_3 moves at speed v_3 in the x-direction with respect to A_2. Calculate the speed at which A_0 observes A_3 to travel.

8A Fluid Dynamics

Explain what is meant by a *material time derivative*, D/Dt. Show that if the material velocity is $\mathbf{u}(\mathbf{x}, t)$ then

$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla.$$

When glass is processed in its liquid state, its temperature, $\theta(\mathbf{x}, t)$, satisfies the equation

$$D\theta/Dt = -\theta.$$

The glass flows in a two-dimensional channel $-1 < y < 1, \quad x > 0$ with steady velocity $\mathbf{u} = (1 - y^2, 0)$. At $x = 0$ the glass temperature is maintained at the constant value θ_0. Find the steady temperature distribution throughout the channel.

9C Optimization

Consider the maximal flow problem on a finite set N, with source A, sink B and capacity constraints c_{ij} for $i, j \in N$. Explain what is meant by a cut and by the capacity of a cut.

Show that the maximal flow value cannot exceed the minimal cut capacity.

Take $N = \{0, 1, 2, 3, 4\}^2$ and suppose that, for $i = (i_1, i_2)$ and $j = (j_1, j_2)$,

$$c_{ij} = \max\{|i_1 - i_2|, |j_1 - j_2|\} \quad \text{if} \quad |i_1 - j_1| + |i_2 - j_2| = 1,$$

and $c_{ij} = 0$ otherwise. Thus the node set is a square grid of 25 points, with positive flow capacity only between nearest neighbours, and where the capacity of an edge in the grid equals the larger of the distances of its two endpoints from the diagonal. Find a maximal flow from $(0, 3)$ to $(3, 0)$. Justify your answer.

SECTION II

10E Linear Algebra

Suppose that V is the set of complex polynomials of degree at most n in the variable x. Find the dimension of V as a complex vector space.

Define

$$e_k : V \to \mathbf{C} \quad \text{by} \quad e_k(\phi) = \frac{d^k\phi}{dx^k}(0).$$

Find a subset of $\{e_k \mid k \in \mathbf{N}\}$ that is a basis of the dual vector space V^*. Find the corresponding dual basis of V.

Define

$$D : V \to V \quad \text{by} \quad D(\phi) = \frac{d\phi}{dx}.$$

Write down the matrix of D with respect to the basis of V that you have just found, and the matrix of the map dual to D with respect to the dual basis.

11E Groups, Rings and Modules

(i) Prove the first Sylow theorem, that a finite group of order $p^n r$ with p prime and p not dividing the integer r has a subgroup of order p^n.

(ii) State the remaining Sylow theorems.

(iii) Show that if p and q are distinct primes then no group of order pq is simple.

12H Geometry

Let $\sigma : V \to U \subset \mathbf{R}^3$ denote a parametrized smooth embedded surface, where V is an open ball in $\mathbf{R}^2$ with coordinates (u, v). Explain briefly the geometric meaning of the *second fundamental form*

$$L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where $L = \sigma_{uu} \cdot \mathbf{N}$, $M = \sigma_{uv} \cdot \mathbf{N}$, $N = \sigma_{vv} \cdot \mathbf{N}$, with $\mathbf{N}$ denoting the unit normal vector to the surface U.

Prove that if the second fundamental form is identically zero, then $\mathbf{N}_u = \mathbf{0} = \mathbf{N}_v$ as vector-valued functions on V, and hence that $\mathbf{N}$ is a constant vector. Deduce that U is then contained in a plane given by $\mathbf{x} \cdot \mathbf{N} = \text{constant}$.

13F Analysis II

For a smooth mapping $F : \mathbf{R}^2 \to \mathbf{R}^2$, the Jacobian $J(F)$ at a point (x, y) is defined as the determinant of the derivative DF, viewed as a linear map $\mathbf{R}^2 \to \mathbf{R}^2$. Suppose that F maps into a curve in the plane, in the sense that F is a composition of two smooth mappings $\mathbf{R}^2 \to \mathbf{R} \to \mathbf{R}^2$. Show that the Jacobian of F is identically zero.

Conversely, let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a smooth mapping whose Jacobian is identically zero. Write $F(x, y) = (f(x, y), g(x, y))$. Suppose that $\partial f/\partial y|_{(0,0)} \neq 0$. Show that $\partial f/\partial y \neq 0$ on some open neighbourhood U of $(0, 0)$ and that on U

$$(\partial g/\partial x,\, \partial g/\partial y) = e(x, y)\,(\partial f/\partial x,\, \partial f/\partial y)$$

for some smooth function e defined on U. Now suppose that $c : \mathbf{R} \to U$ is a smooth curve of the form $t \mapsto (t, \alpha(t))$ such that $F \circ c$ is constant. Write down a differential equation satisfied by α. Apply an existence theorem for differential equations to show that there is a neighbourhood V of $(0, 0)$ such that every point in V lies on a curve $t \mapsto (t, \alpha(t))$ on which F is constant.

[*A function is said to be smooth when it is infinitely differentiable. Detailed justification of the smoothness of the functions in question is not expected.*]

14D Complex Analysis or Complex Methods

Let Ω be the region enclosed between the two circles C_1 and C_2, where

$$C_1 = \{z \in \mathbf{C} : |z - i| = 1\}, \qquad C_2 = \{z \in \mathbf{C} : |z - 2i| = 2\}.$$

Find a conformal mapping that maps Ω onto the unit disc.

[*Hint: you may find it helpful first to map Ω to a strip in the complex plane.*]

15G Methods

Verify that $y = e^{-x}$ is a solution of the differential equation

$$(x+2)y'' + (x+1)y' - y = 0,$$

and find a second solution of the form $ax + b$.

Let L be the operator

$$L[y] = y'' + \frac{(x+1)}{(x+2)}y' - \frac{1}{(x+2)}y$$

on functions $y(x)$ satisfying

$$y'(0) = y(0) \quad \text{and} \quad \lim_{x\to\infty} y(x) = 0.$$

The Green's function $G(x,\xi)$ for L satisfies

$$L[G] = \delta(x-\xi),$$

with $\xi > 0$. Show that

$$G(x,\xi) = -\frac{(\xi+1)}{(\xi+2)}e^{\xi - x}$$

for $x > \xi$, and find $G(x,\xi)$ for $x < \xi$.

Hence or otherwise find the solution of

$$L[y] = -(x+2)e^{-x},$$

for $x \geqslant 0$, with $y(x)$ satisfying the boundary conditions above.

16B Quantum Mechanics

The spherically symmetric bound state wavefunctions $\psi(r)$, where $r = |\mathbf{x}|$, for an electron orbiting in the Coulomb potential $V(r) = -e^2/(4\pi\epsilon_0 r)$ of a hydrogen atom nucleus, can be modelled as solutions to the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + \frac{a}{r}\psi(r) - b^2\psi(r) = 0$$

for $r \geqslant 0$, where $a = e^2 m/(2\pi\epsilon_0\hbar^2)$, $b = \sqrt{-2mE}/\hbar$, and E is the energy of the corresponding state. Show that there are normalisable and continuous wavefunctions $\psi(r)$ satisfying this equation with energies

$$E = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 N^2}$$

for all integers $N \geqslant 1$.

17G Electromagnetism

Derive from Maxwell's equations the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'$$

giving the magnetic field $\mathbf{B}(\mathbf{r})$ produced by a steady current density $\mathbf{j}(\mathbf{r})$ that vanishes outside a bounded region V.

[*You may assume that the divergence of the magnetic vector potential is zero.*]

A steady current density $\mathbf{j}(\mathbf{r})$ has the form $\mathbf{j} = (0, j_\phi(\mathbf{r}), 0)$ in cylindrical polar coordinates (r, ϕ, z) where

$$j_\phi(\mathbf{r}) = \begin{cases} kr & 0 \leqslant r \leqslant b, \quad -h \leqslant z \leqslant h, \\ 0 & \text{otherwise,} \end{cases}$$

and k is a constant. Find the magnitude and direction of the magnetic field at the origin.

$$\left[\textit{Hint}: \quad \int_{-h}^{h} \frac{dz}{(r^2+z^2)^{3/2}} = \frac{2h}{r^2(h^2+r^2)^{1/2}} \, . \right]$$

18D Numerical Analysis

(a) For a positive weight function w, let

$$\int_{-1}^{1} f(x)w(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

be the corresponding Gaussian quadrature with $n+1$ nodes. Prove that all the coefficients a_i are positive.

(b) The integral

$$I(f) = \int_{-1}^{1} f(x)w(x)\,dx$$

is approximated by a quadrature

$$I_n(f) = \sum_{i=0}^{n} a_i^{(n)} f(x_i^{(n)})$$

which is exact on polynomials of degree $\leqslant n$ and has positive coefficients $a_i^{(n)}$. Prove that, for any f continuous on $[-1,1]$, the quadrature converges to the integral, i.e.,

$$|I(f) - I_n(f)| \to 0 \quad \text{as} \quad n \to \infty.$$

[*You may use the Weierstrass theorem: for any f continuous on $[-1,1]$, and for any $\epsilon > 0$, there exists a polynomial Q of degree $n = n(\epsilon, f)$ such that* $\max_{x\in[-1,1]} |f(x) - Q(x)| < \epsilon$.]

19C Statistics

Suppose that $X_1, \ldots, X_n$ are independent normal random variables of unknown mean θ and variance 1. It is desired to test the hypothesis $H_0 : \theta \leq 0$ against the alternative $H_1 : \theta > 0$. Show that there is a uniformly most powerful test of size $\alpha = 1/20$ and identify a critical region for such a test in the case $n = 9$. If you appeal to any theoretical result from the course you should also prove it.

[*The 95th percentile of the standard normal distribution is 1.65.*]

20C Markov Chains

Consider the Markov chain $(X_n)_{n\geq 0}$ on the integers $\mathbb{Z}$ whose non-zero transition probabilities are given by $p_{0,1} = p_{0,-1} = 1/2$ and

$$p_{n,n-1} = 1/3, \quad p_{n,n+1} = 2/3, \qquad \text{for } n \geq 1,$$

$$p_{n,n-1} = 3/4, \quad p_{n,n+1} = 1/4, \qquad \text{for } n \leqslant -1.$$

(a) Show that, if $X_0 = 1$, then $(X_n)_{n\geq 0}$ hits 0 with probability 1/2.

(b) Suppose now that $X_0 = 0$. Show that, with probability 1, as $n \to \infty$ either $X_n \to \infty$ or $X_n \to -\infty$.

(c) In the case $X_0 = 0$ compute $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$.

END OF PAPER