UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 6th June, 2006 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **all four** *questions from Section I and* **at most five** *questions from Section II. In Section II, no more than* **three** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **A** *and* **D** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Algebra and Geometry

Give an example of a real 3×3 matrix A with eigenvalues -1, $(1 \pm i)/\sqrt{2}$. Prove or give a counterexample to the following statements:

(i) any such A is diagonalisable over $\mathbb{C}$;

(ii) any such A is orthogonal;

(iii) any such A is diagonalisable over $\mathbb{R}$.

2D Algebra and Geometry

Show that if H and K are subgroups of a group G, then $H \cap K$ is also a subgroup of G. Show also that if H and K have orders p and q respectively, where p and q are coprime, then $H \cap K$ contains only the identity element of G. [*You may use Lagrange's theorem provided it is clearly stated.*]

3A Vector Calculus

Consider the vector field $\mathbf{F}(\mathbf{x}) = \big((3x^3 - x^2)y,\ (y^3 - 2y^2 + y)x,\ z^2 - 1\big)$ and let S be the surface of a unit cube with one corner at (0, 0, 0), another corner at (1, 1, 1) and aligned with edges along the x-, y- and z-axes. Use the divergence theorem to evaluate

$$I = \int_S \mathbf{F} \cdot d\mathbf{S}\,.$$

Verify your result by calculating the integral directly.

4A Vector Calculus

Use suffix notation in Cartesian coordinates to establish the following two identities for the vector field $\mathbf{v}$:

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0\,, \qquad (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla(\tfrac{1}{2}|\mathbf{v}|^2) - \mathbf{v} \times (\nabla \times \mathbf{v})\,.$$

SECTION II

5D Algebra and Geometry

Let G be a group and let A be a non-empty subset of G. Show that

$$C(A) = \{g \in G : \ gh = hg \quad \text{for all} \ \ h \in A\}$$

is a subgroup of G.

Show that $\rho : G \times G \to G$ given by

$$\rho(g,h) = ghg^{-1}$$

defines an action of G on itself.

Suppose G is finite, let $O_1, \ldots, O_n$ be the orbits of the action ρ and let $h_i \in O_i$ for $i = 1, \ldots, n$. Using the Orbit–Stabilizer Theorem, or otherwise, show that

$$|G| = |C(G)| + \sum_i |G|/|C(\{h_i\})|$$

where the sum runs over all values of i such that $|O_i| > 1$.

Let G be a finite group of order p^r, where p is a prime and r is a positive integer. Show that $C(G)$ contains more than one element.

6D Algebra and Geometry

Let $\theta : G \to H$ be a homomorphism between two groups G and H. Show that the image of θ, $\theta(G)$, is a subgroup of H; show also that the kernel of θ, $\ker(\theta)$, is a normal subgroup of G.

Show that $G/\ker(\theta)$ is isomorphic to $\theta(G)$.

Let $O(3)$ be the group of 3×3 real orthogonal matrices and let $SO(3) \subset O(3)$ be the set of orthogonal matrices with determinant 1. Show that $SO(3)$ is a normal subgroup of $O(3)$ and that $O(3)/SO(3)$ is isomorphic to the cyclic group of order 2.

Give an example of a homomorphism from $O(3)$ to $SO(3)$ with kernel of order 2.

7D Algebra and Geometry

Let $SL(2,\mathbb{R})$ be the group of 2×2 real matrices with determinant 1 and let $\sigma : \mathbb{R} \to SL(2,\mathbb{R})$ be a homomorphism. On $K = \mathbb{R} \times \mathbb{R}^2$ consider the product

$$(x, \mathbf{v}) * (y, \mathbf{w}) = (x + y, \mathbf{v} + \sigma(x)\mathbf{w}).$$

Show that K with this product is a group.

Find the homomorphism or homomorphisms σ for which K is a commutative group.

Show that the homomorphisms σ for which the elements of the form $(0, \mathbf{v})$ with $\mathbf{v} = (a, 0)$, $a \in \mathbb{R}$, commute with every element of K are precisely those such that

$$\sigma(x) = \begin{pmatrix} 1 & r(x) \\ 0 & 1 \end{pmatrix},$$

with $r : (\mathbb{R}, +) \to (\mathbb{R}, +)$ an arbitrary homomorphism.

8D Algebra and Geometry

Show that every Möbius transformation can be expressed as a composition of maps of the forms: $S_1(z) = z + \alpha$, $S_2(z) = \lambda\, z$ and $S_3(z) = 1/z$, where $\alpha, \lambda \in \mathbb{C}$.

Show that if z_1, z_2, z_3 and w_1, w_2, w_3 are two triples of distinct points in $\mathbb{C} \cup \{\infty\}$, there exists a unique Möbius transformation that takes z_j to w_j $(j = 1, 2, 3)$.

Let G be the group of those Möbius transformations which map the set $\{0, 1, \infty\}$ to itself. Find all the elements of G. To which standard group is G isomorphic?

9A Vector Calculus

Evaluate the line integral

$$\int \alpha(x^2 + xy)dx + \beta(x^2 + y^2)dy,$$

with α and β constants, along each of the following paths between the points $A = (1, 0)$ and $B = (0, 1)$:

(i) the straight line between A and B;

(ii) the x-axis from A to the origin $(0, 0)$ followed by the y-axis to B;

(iii) anti-clockwise from A to B around the circular path centred at the origin $(0, 0)$.

You should obtain the same answer for the three paths when $\alpha = 2\beta$. Show that when $\alpha = 2\beta$, the integral takes the same value along *any* path between A and B.

10A Vector Calculus

State Stokes' theorem for a vector field $\mathbf{A}$.

By applying Stokes' theorem to the vector field $\mathbf{A} = \phi\,\mathbf{k}$, where $\mathbf{k}$ is an arbitrary constant vector in $\mathbb{R}^3$ and ϕ is a scalar field defined on a surface S bounded by a curve ∂S, show that

$$\int_S d\mathbf{S} \times \nabla\phi = \int_{\partial S} \phi \; d\mathbf{x}\,.$$

For the vector field $\mathbf{A} = x^2 y^4 (1,\, 1,\, 1)$ in Cartesian coordinates, evaluate the line integral

$$I = \int \mathbf{A} \cdot d\mathbf{x}\,,$$

around the boundary of the quadrant of the unit circle lying between the x- and y-axes, that is, along the straight line from (0, 0, 0) to (1, 0, 0), then the circular arc $x^2 + y^2 = 1,\ z = 0$ from (1, 0, 0) to (0, 1, 0) and finally the straight line from (0, 1, 0) back to (0, 0, 0).

11A Vector Calculus

In a region R of $\mathbb{R}^3$ bounded by a closed surface S, suppose that ϕ_1 and ϕ_2 are both solutions of $\nabla^2\phi = 0$, satisfying boundary conditions on S given by $\phi = f$ on S, where f is a given function. Prove that $\phi_1 = \phi_2$.

In $\mathbb{R}^2$ show that

$$\phi(x,y) = (a_1 \cosh \lambda x + a_2 \sinh \lambda x)(b_1 \cos \lambda y + b_2 \sin \lambda y)$$

is a solution of $\nabla^2\phi = 0$, for any constants a_1, a_2, b_1, b_2 and λ. Hence, or otherwise, find a solution $\phi(x,y)$ in the region $x \geqslant 0$ and $0 \leqslant y \leqslant a$ which satisfies:

$$\begin{aligned}
&\phi(x,0) = 0\,, \quad \phi(x,a) = 0, \quad x \geqslant 0\,, \\
&\phi(0,y) = \sin\frac{n\pi y}{a}\,, \quad \phi(x,y) \to 0 \ \text{ as } \ x \to \infty\,, \quad 0 \leqslant y \leqslant a\,,
\end{aligned}$$

where a is a real constant and n is an integer.

12A Vector Calculus

Define what is meant by an isotropic tensor. By considering a rotation of a second rank isotropic tensor B_{ij} by $90°$ about the z-axis, show that its components must satisfy $B_{11} = B_{22}$ and $B_{13} = B_{31} = B_{23} = B_{32} = 0$. Now consider a second and different rotation to show that B_{ij} must be a multiple of the Kronecker delta, δ_{ij}.

Suppose that a homogeneous but anisotropic crystal has the conductivity tensor

$$\sigma_{ij} = \alpha\delta_{ij} + \gamma n_i n_j \,,$$

where α, γ are real constants and the n_i are the components of a constant unit vector $\mathbf{n}$ ($\mathbf{n} \cdot \mathbf{n} = 1$). The electric current density $\mathbf{J}$ is then given in components by

$$J_i = \sigma_{ij} E_j,$$

where E_j are the components of the electric field $\mathbf{E}$. Show that

(i) if $\alpha \neq 0$ and $\gamma \neq 0$, then there is a plane such that if $\mathbf{E}$ lies in this plane, then $\mathbf{E}$ and $\mathbf{J}$ must be parallel, and

(ii) if $\gamma \neq -\alpha$ and $\alpha \neq 0$, then $\mathbf{E} \neq 0$ implies $\mathbf{J} \neq 0$.

If $D_{ij} = \epsilon_{ijk} n_k$, find the value of γ such that

$$\sigma_{ij} D_{jk} D_{km} = -\sigma_{im} \,.$$

END OF PAPER