UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 2nd June, 2006 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **all four** *questions from Section I and* **at most five** *questions from Section II. In Section II, no more than* **three** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **B** *and* **F** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

Solve the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = x_0,$$

and sketch the phase portrait. Describe the behaviour as $t \to +\infty$ and as $t \to -\infty$ of solutions with initial value satisfying $0 < x_0 < 1$.

2B Differential Equations

Consider the first order system

$$\frac{d\mathbf{x}}{dt} - A\mathbf{x} = e^{\lambda t}\mathbf{v}$$

to be solved for $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n$, where A is an $n \times n$ matrix, $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$. Show that if λ is not an eigenvalue of A there is a solution of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}$. For $n = 2$, given

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \lambda = 1, \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

find this solution.

3F Probability

What is a convex function? State Jensen's inequality for a convex function of a random variable which takes finitely many values.

Let $p \geqslant 1$. By using Jensen's inequality, or otherwise, find the smallest constant c_p so that

$$(a+b)^p \leqslant c_p\,(a^p + b^p) \quad \text{for all } a, b \geqslant 0.$$

[*You may assume that* $x \mapsto |x|^p$ *is convex for* $p \geqslant 1$.]

4F Probability

Let K be a fixed positive integer and X a discrete random variable with values in $\{1, 2, \ldots, K\}$. Define the *probability generating function* of X. Express the mean of X in terms of its probability generating function. The *Dirichlet probability generating function* of X is defined as

$$q(z) = \sum_{n=1}^{K} \frac{1}{n^z} P(X = n).$$

Express the mean of X and the mean of $\log X$ in terms of $q(z)$.

SECTION II

5B Differential Equations

Find the general solution of the system

$$\frac{dx}{dt} = 5x + 3y + e^{2t},$$
$$\frac{dy}{dt} = 2x + 2e^{t},$$
$$\frac{dz}{dt} = x + y + e^{t}.$$

6B Differential Equations

(i) Consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + f(t,x)$$

and, using the change of variables $(t,x) \mapsto (s,y) = (t, x-t)$, show that it can be transformed into an equation of the form

$$\frac{\partial U}{\partial s} = \frac{\partial^2 U}{\partial y^2} + F(s,y)$$

where $U(s,y) = u(s, y+s)$ and you should determine $F(s,y)$.

(ii) Let $H(y)$ be the Heaviside function. Find the general continuously differentiable solution of the equation

$$w''(y) + H(y) = 0.$$

(iii) Using (i) and (ii), find a continuously differentiable solution of

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + H(x-t)$$

such that $u(t,x) \to 0$ as $x \to -\infty$ and $u(t,x) \to -\infty$ as $x \to +\infty$.

7B Differential Equations

Let p, q be continuous functions and let $y_1(x)$ and $y_2(x)$ be, respectively, the solutions of the initial value problems

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \quad y_1(0) = 0,\ y_1'(0) = 1,$$

$$y_2'' + p(x)y_2' + q(x)y_2 = 0, \quad y_2(0) = 1,\ y_2'(0) = 0.$$

If f is any continuous function show that the solution of

$$y'' + p(x)y' + q(x)y = f(x), \quad y(0) = 0,\ y'(0) = 0$$

is

$$y(x) = \int_0^x \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{W(s)} f(s)ds,$$

where $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is the Wronskian. Use this method to find $y = y(x)$ such that

$$y'' + y = \sin x, \quad y(0) = 0,\ y'(0) = 0.$$

8B Differential Equations

Obtain a power series solution of the problem

$$xy'' + y = 0, \quad y(0) = 0,\ y'(0) = 1.$$

[*You need not find the general power series solution.*]

Let $y_0(x), y_1(x), y_2(x), \ldots$ be defined recursively as follows: $y_0(x) = x$. Given $y_{n-1}(x)$, define $y_n(x)$ to be the solution of

$$xy_n''(x) = -y_{n-1}, \quad y_n(0) = 0,\ y_n'(0) = 1.$$

By calculating y_1, y_2, y_3, or otherwise, obtain and prove a general formula for $y_n(x)$. Comment on the relation to the power series solution obtained previously.

9F Probability

Suppose that a population evolves in generations. Let Z_n be the number of members in the n-th generation and $Z_0 \equiv 1$. Each member of the n-th generation gives birth to a family, possibly empty, of members of the $(n+1)$-th generation; the size of this family is a random variable and we assume that the family sizes of all individuals form a collection of independent identically distributed random variables with the same generating function G.

Let G_n be the generating function of Z_n. State and prove a formula for G_n in terms of G. Use this to compute the variance of Z_n.

Now consider the *total* number of individuals in the first n generations; this number is a random variable and we write H_n for its generating function. Find a formula that expresses $H_{n+1}(s)$ in terms of $H_n(s)$, $G(s)$ and s.

10F Probability

Let X, Y be independent random variables with values in $(0, \infty)$ and the same probability density $\frac{2}{\sqrt{\pi}} e^{-x^2}$. Let $U = X^2 + Y^2$, $V = Y/X$. Compute the joint probability density of U, V and the marginal densities of U and V respectively. Are U and V independent?

11F Probability

A normal deck of playing cards contains 52 cards, four each with face values in the set $\mathcal{F} = \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$. Suppose the deck is well shuffled so that each arrangement is equally likely. Write down the probability that the top and bottom cards have the same face value.

Consider the following algorithm for shuffling:

S1: Permute the deck randomly so that each arrangement is equally likely.

S2: If the top and bottom cards do *not* have the same face value, toss a biased coin that comes up heads with probability p and go back to step S1 if head turns up. Otherwise stop.

All coin tosses and all permutations are assumed to be independent. When the algorithm stops, let X and Y denote the respective face values of the top and bottom cards and compute the probability that $X = Y$. Write down the probability that $X = x$ for some $x \in \mathcal{F}$ and the probability that $Y = y$ for some $y \in \mathcal{F}$. What value of p will make X and Y independent random variables? Justify your answer.

12F Probability

Let $\gamma > 0$ and define

$$f(x) = \gamma \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Find γ such that f is a probability density function. Let $\{X_i : i \geqslant 1\}$ be a sequence of independent, identically distributed random variables, each having f with the correct choice of γ as probability density. Compute the probability density function of $X_1 + \cdots + X_n$. [*You may use the identity*

$$m \int_{-\infty}^{\infty} \left\{ \left(1+y^2\right) \left[m^2 + (x-y)^2\right] \right\}^{-1} dy = \pi (m+1) \left\{ (m+1)^2 + x^2 \right\}^{-1},$$

valid for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$.]

Deduce the probability density function of

$$\frac{X_1 + \cdots + X_n}{n}.$$

Explain why your result does not contradict the weak law of large numbers.

END OF PAPER