MATHEMATICAL TRIPOS Part II 2005

List of Courses

1/I/1H Number Theory

Define the Legendre symbol $\left(\frac{a}{p}\right)$. Prove that, if p is an odd prime, then

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

Use the law of quadratic reciprocity to calculate $\left(\frac{91}{167}\right)$.

[*You may use the Gauss Lemma without proof.*]

2/I/1H Number Theory

Recall that, if p is an odd prime, a *primitive root* modulo p is a generator of the cyclic (multiplicative) group $(\mathbb{Z}/p\mathbb{Z})^{\times}$. Let p be an odd prime of the form $2^{2^n}+1$; show that a is a primitive root mod p if and only if a is not a quadratic residue mod p. Use this result to prove that 7 is a primitive root modulo every such prime.

3/I/1H Number Theory

Let $\pi(x)$ be the number of primes $p \leqslant x$. State the Legendre formula, and prove that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0.$$

[*You may use the formula*

$$\prod_{p\leqslant x} (1-1/p)^{-1} \geqslant \log x$$

without proof.]

3/II/11H Number Theory

Show that there are exactly two reduced positive definite integer binary quadratic forms with discriminant -20; write these forms down.

State a criterion for an odd integer n to be properly represented by a positive definite integer binary quadratic form of given discriminant d.

Describe, in terms of congruences modulo 20, which primes other than $2, 5$ are properly represented by the form x^2+5y^2, and justify your answer.

4/I/1H Number Theory

If n is an odd integer and b is an integer prime with n, state what it means for n to be a pseudoprime to the base b. What is a Carmichael number? State a criterion for n to be a Carmichael number and use the criterion to show that:

(i) Every Carmichael number is the product of at least three distinct primes.

(ii) 561 is a Carmichael number.

4/II/11H Number Theory

(a) Let N be a non-square integer. Describe the integer solutions of the Pell equation $x^2 - Ny^2 = 1$ in terms of the convergents to $\sqrt{N}$. Show that the set of integer solutions forms an abelian group. Denote the addition law in this group by $\circ$; given solutions (x_0, y_0) and (x_1, y_1), write down an explicit formula for $(x_0, y_0) \circ (x_1, y_1)$. If (x, y) is a solution, write down an explicit formula for $(x, y) \circ (x, y) \circ (x, y)$ in the group law.

(b) Find the continued fraction expansion of $\sqrt{11}$. Find the smallest solution in integers $x, y > 0$ of the Pell equation $x^2 - 11y^2 = 1$. Use the formula in Part (a) to compute $(x, y) \circ (x, y) \circ (x, y)$.

1/I/2F Topics in Analysis

Prove that $\cosh(1/2)$ is irrational.

1/II/11F Topics in Analysis

State and prove a discrete form of Brouwer's theorem, concerning colourings of points in triangular grids. Use it to deduce that there is no continuous retraction from a disc to its boundary.

2/I/2F Topics in Analysis

(i) Let α be an algebraic number and let p and q be integers with $q \neq 0$. What does Liouville's theorem say about α and p/q?

(ii) Let p and q be integers with $q \neq 0$. Prove that

$$\left|\sqrt{2} - \frac{p}{q}\right| \geqslant \frac{1}{4q^2} \ .$$

[*In (ii), you may not use Liouville's theorem unless you prove it.*]

2/II/11F Topics in Analysis

(i) State the Baire category theorem. Deduce from it a statement about nowhere dense sets.

(ii) Let X be the set of all real numbers with decimal expansions consisting of the digits 4 and 5 only. Prove that there is a real number t that cannot be written in the form $x + y$ with $x \in X$ and y rational.

3/I/2F Topics in Analysis

Let $-1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$ and let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n} a_i p(x_i)$$

for every polynomial p of degree less than $2n$. Prove the following three facts.

(i) $a_i > 0$ for every i.

(ii) $\sum_{i=1}^{n} a_i = 2$.

(iii) The numbers $x_1, x_2, \ldots, x_n$ are the roots of the Legendre polynomial of degree n.

[*You may assume standard orthogonality properties of the Legendre polynomials.*]

4/I/2F Topics in Analysis

(i) Let $D \subset \mathbb{C}$ be a domain, let $f : D \to \mathbb{C}$ be an analytic function and let $z_0 \in D$. What does Taylor's theorem say about z_0, f and D?

(ii) Let K be the square consisting of all complex numbers z such that

$$-1 \leqslant \operatorname{Re}(z) \leqslant 1 \text{ and } -1 \leqslant \operatorname{Im}(z) \leqslant 1\,,$$

and let w be a complex number not belonging to K. Prove that the function $f(z) = (z - w)^{-1}$ can be uniformly approximated on K by polynomials.

1/I/3G **Geometry of Group Actions**

Let G be a subgroup of the group of isometries $\mathrm{Isom}(\mathbb{R}^2)$ of the Euclidean plane. What does it mean to say that G is *discrete*?

Supposing that G is discrete, show that the subgroup G_T of G consisting of all translations in G is generated by translations in at most two linearly independent vectors in $\mathbb{R}^2$. Show that there is a homomorphism $G \to O(2)$ with kernel G_T.

Draw, and briefly explain, pictures which illustrate two different possibilities for G when G_T is isomorphic to the additive group $\mathbb{Z}$.

1/II/12G **Geometry of Group Actions**

What is the *limit set* of a subgroup G of Möbius transformations?

Suppose that G is complicated and has no finite orbit in $\mathbb{C} \cup \{\infty\}$. Prove that the limit set of G is infinite. Can the limit set be countable?

State Jørgensen's inequality, and deduce that not every two-generator subgroup $G = \langle A, B \rangle$ of Möbius transformations is discrete. Briefly describe two examples of discrete two-generator subgroups, one for which the limit set is connected and one for which it is disconnected.

2/I/3G **Geometry of Group Actions**

Describe the *geodesics* in the disc model of the hyperbolic plane $\mathbb{H}^2$.

Define the *area* of a region in $\mathbb{H}^2$. Compute the area $A(r)$ of a hyperbolic circle of radius r from the definition just given. Compute the circumference $C(r)$ of a hyperbolic circle of radius r, and check explicitly that $dA(r)/dr = C(r)$.

How could you define π geometrically if you lived in $\mathbb{H}^2$? Briefly justify your answer.

3/I/3G **Geometry of Group Actions**

By considering fixed points in $\mathbb{C} \cup \{\infty\}$, prove that any complex Möbius transformation is conjugate either to a map of the form $z \mapsto kz$ for some $k \in \mathbb{C}$ or to $z \mapsto z + 1$. Deduce that two Möbius transformations g,h (neither the identity) are conjugate if and only if $\mathrm{tr}^2(g) = \mathrm{tr}^2(h)$.

Does every Möbius transformation g also have a fixed point in $\mathbb{H}^3$? Briefly justify your answer.

4/I/3G Geometry of Group Actions

Show that a set $F \subset \mathbb{R}^n$ with Hausdorff dimension strictly less than one is totally disconnected.

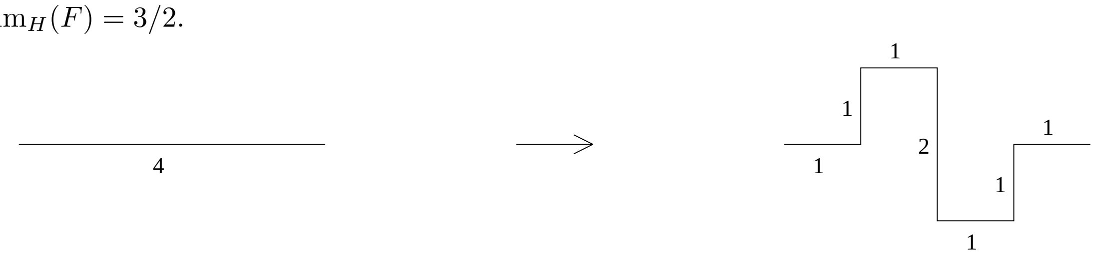
What does it mean for a Möbius transformation to *pair two discs*? By considering a pair of disjoint discs and a pair of tangent discs, or otherwise, explain in words why there is a 2-generator Schottky group with limit set $\Lambda \subset \mathbb{S}^2$ which has Hausdorff dimension at least 1 but which is not homeomorphic to a circle.

4/II/12G Geometry of Group Actions

For real $s \geqslant 0$ and $F \subset \mathbb{R}^n$, give a careful definition of the s-dimensional *Hausdorff measure* of F and of the *Hausdorff dimension* $\dim_H(F)$ of F.

For $1 \leqslant i \leqslant k$, suppose $S_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with contraction factor $c_i \in (0,1)$. Prove there is a unique non-empty compact invariant set I for the $\{S_i\}$. State a formula for the Hausdorff dimension of I, under an assumption on the S_i you should state.

Hence show the Hausdorff dimension of the fractal F given by iterating the scheme below (at each stage replacing each edge by a new copy of the generating template) is $\dim_H(F) = 3/2$.



[Numbers denote lengths]

1/I/4J Coding and Cryptography

Briefly describe the methods of Shannon-Fano and Huffman for economical coding. Illustrate both methods by finding decipherable binary codings in the case where messages $\mu_1, \ldots, \mu_5$ are emitted with probabilities $0.45, 0.25, 0.2, 0.05, 0.05$. Compute the expected word length in each case.

2/I/4J Coding and Cryptography

What is a *linear binary code*? What is the *weight* $w(C)$ of a linear binary code C? Define the bar product $C_1|C_2$ of two binary linear codes C_1 and C_2, stating the conditions that C_1 and C_2 must satisfy. Under these conditions show that

$$w(C_1|C_2) \geqslant \min(2w(C_1), w(C_2)).$$

2/II/12J Coding and Cryptography

What does it means to say that $f : \mathbb{F}_2^d \to \mathbb{F}_2^d$ is a linear feedback shift register? Let $(x_n)_{n\geqslant 0}$ be a stream produced by such a register. Show that there exist N, M with $N + M \leqslant 2^d - 1$ such that $x_{r+N} = x_r$ for all $r \geqslant M$.

Explain and justify the Berlekamp–Massey method for 'breaking' a cipher stream arising from a linear feedback register of unknown length.

Let x_n, y_n, z_n be three streams produced by linear feedback registers. Set

$$k_n = x_n \text{ if } y_n = z_n$$

$$k_n = y_n \text{ if } y_n \neq z_n\,.$$

Show that k_n is also a stream produced by a linear feedback register. Sketch proofs of any theorems that you use.

3/I/4J Coding and Cryptography

Briefly explain how and why a signature scheme is used. Describe the el Gamal scheme.

3/II/12J **Coding and Cryptography**

Define a *cyclic code*. Define the generator and check polynomials of a cyclic code and show that they exist.

Show that Hamming's original code is a cyclic code with check polynomial $X^4 + X^2 + X + 1$. What is its generator polynomial? Does Hamming's original code contain a subcode equivalent to its dual?

4/I/4J **Coding and Cryptography**

What does it mean to transmit reliably at rate r through a binary symmetric channel (BSC) with error probability p? Assuming Shannon's second coding theorem, compute the supremum of all possible reliable transmission rates of a BSC. What happens if (i) p is very small, (ii) $p = 1/2$, or (iii) $p > 1/2$?

1/I/5I **Statistical Modelling**

Suppose that $Y_1, \ldots, Y_n$ are independent random variables, and that Y_i has probability density function

$$f(y_i|\theta_i, \phi) = \exp\left[\frac{(y_i\theta_i - b(\theta_i))}{\phi} + c(y_i, \phi)\right] .$$

Assume that $\mathbb{E}(Y_i) = \mu_i$ and that there is a known link function $g(.)$ such that

$$g(\mu_i) = \beta^T x_i \ ,$$

where $x_1, \ldots, x_n$ are known p-dimensional vectors and β is an unknown p-dimensional parameter. Show that $\mathbb{E}(Y_i) = b'(\theta_i)$ and that, if $\ell(\beta, \phi)$ is the log-likelihood function from the observations $(y_1, \ldots, y_n)$, then

$$\frac{\partial \ell(\beta, \phi)}{\partial \beta} = \sum_1^n \frac{(y_i - \mu_i)x_i}{g'(\mu_i)V_i} \ ,$$

where V_i is to be defined.

1/II/13I **Statistical Modelling**

The Independent, June 1999, under the headline 'Tourists get hidden costs warnings' gave the following table of prices in pounds, called 'How the resorts compared'.

Algarve	8.00	0.50	3.50	3.00	4.00	100.00
CostaDelSol	6.95	1.30	4.10	12.30	4.10	130.85
Majorca	10.25	1.45	5.35	6.15	3.30	122.20
Tenerife	12.30	1.25	4.90	3.70	2.90	130.85
Florida	15.60	1.90	5.05	5.00	2.50	114.00
Tunisia	10.90	1.40	5.45	1.90	2.75	218.10
Cyprus	11.60	1.20	5.95	3.00	3.60	149.45
Turkey	6.50	1.05	6.50	4.90	2.85	263.00
Corfu	5.20	1.05	3.75	4.20	2.50	137.60
Sorrento	7.70	1.40	6.30	8.75	4.75	215.40
Malta	11.20	0.70	4.55	8.00	4.80	87.85
Rhodes	6.30	1.05	5.20	3.15	2.70	261.30
Sicily	13.25	1.75	4.20	7.00	3.85	174.40
Madeira	10.25	0.70	5.10	6.85	6.85	153.70

Here the column headings are, respectively: Three-course meal, Bottle of Beer, Suntan Lotion, Taxi (5km), Film (24 exp), Car Hire (per week). Interpret the R commands, and explain how to interpret the corresponding (slightly abbreviated) R output given below. Your solution should include a careful statement of the underlying statistical model, but you may quote without proof any distributional results required.

```
> price = scan("dresorts") ; price

> Goods = gl(6,1,length=84); Resort=gl(14,6,length=84)

> first.lm = lm(log(price) ~ Goods + Resort)

> summary(first.lm)

 Coefficients:

               Estimate Std. Error t value Pr(>|t|)

(Intercept)   1.8778     0.1629  11.527  < 2e-16

Goods2       -2.1084     0.1295 -16.286  < 2e-16

Goods3       -0.6343     0.1295  -4.900 6.69e-06

Goods4       -0.6284     0.1295  -4.854 7.92e-06

Goods5       -0.9679     0.1295  -7.476 2.49e-10

Goods6        2.8016     0.1295  21.640  < 2e-16

Resort2       0.4463     0.1978   2.257  0.02740

Resort3       0.4105     0.1978   2.076  0.04189

Resort4       0.3067     0.1978   1.551  0.12584

Resort5       0.4235     0.1978   2.142  0.03597

Resort6       0.2883     0.1978   1.458  0.14963

Resort7       0.3457     0.1978   1.748  0.08519

Resort8       0.3787     0.1978   1.915  0.05993

Resort9       0.0943     0.1978   0.477  0.63508

Resort10      0.5981     0.1978   3.025  0.00356

Resort11      0.3281     0.1978   1.659  0.10187

Resort12      0.2525     0.1978   1.277  0.20616

Resort13      0.5508     0.1978   2.785  0.00700

Resort14      0.4590     0.1978   2.321  0.02343


 Residual standard error: 0.3425 on 65 degrees of freedom

 Multiple R-Squared: 0.962
```

2/I/5I Statistical Modelling

You see below three R commands, and the corresponding output (which is slightly abbreviated). Explain the effects of the commands. How is the deviance defined, and why do we have d.f.=7 in this case? Interpret the numerical values found in the output.

```
> n = scan()
  3 5 16 12 11 34 37 51 56

> i = scan ()
  1 2 3 4 5 6 7 8 9

> summary(glm(n~i,poisson))
  deviance = 13.218
      d.f. = 7
  Coefficients:
                 Value     Std.Error
  (intercept)    1.363     0.2210
  i              0.3106    0.0382
```

3/I/5I Statistical Modelling

Consider the model $Y = X\beta + \epsilon$, where Y is an n-dimensional observation vector, X is an $n \times p$ matrix of rank p, ϵ is an n-dimensional vector with components $\epsilon_1, \ldots, \epsilon_n$, and $\epsilon_1, \ldots, \epsilon_n$ are independently and normally distributed, each with mean 0 and variance σ^2.

(a) Let $\hat{\beta}$ be the least-squares estimator of β. Show that

$$(X^T X)\hat{\beta} = X^T Y$$

and find the distribution of $\hat{\beta}$.

(b) Define $\hat{Y} = X\hat{\beta}$. Show that $\hat{Y}$ has distribution $N(X\beta, \sigma^2 H)$, where H is a matrix that you should define.

[*You may quote without proof any results you require about the multivariate normal distribution.*]

4/I/5I **Statistical Modelling**

You see below five R commands, and the corresponding output (which is slightly abbreviated). Without giving any mathematical proofs, explain the purpose of these commands, and interpret the output.

```
> Yes = c(12, 27,11,24)

> Total = c(117,170,52,118)

> Sclass = c("a","a","b","b")

> Sclass = factor(Sclass)

> summary(glm(Yes/Total~ Sclass, binomial, weights=Total))

Coefficients:

            Estimate Std. Error z value

(Intercept)  -1.8499     0.1723 -10.739

Sclassb       0.4999     0.2562   1.951


Residual deviance: 1.9369  on 2  degrees of freedom


Number of Fisher Scoring iterations: 4
```

4/II/13I **Statistical Modelling**

(i) Suppose that $Y_1, \ldots, Y_n$ are independent random variables, and that Y_i has probability density function

$$f(y_i|\beta,\nu) = \left(\frac{\nu y_i}{\mu_i}\right)^{\nu} e^{-y_i\nu/\mu_i} \frac{1}{\Gamma(\nu)} \frac{1}{y_i} \quad \text{for } y_i > 0$$

where

$$1/\mu_i = \beta^T x_i \ , \text{ for } \quad 1 \leqslant i \leqslant n,$$

and $x_1, \ldots, x_n$ are given p-dimensional vectors, and ν is known.

Show that $\mathbb{E}(Y_i) = \mu_i$ and that $\operatorname{var}(Y_i) = \mu_i^2/\nu$.

(ii) Find the equation for $\hat{\beta}$, the maximum likelihood estimator of β, and suggest an iterative scheme for its solution.

(iii) If $p = 2$, and $x_i = \begin{pmatrix} 1 \\ z_i \end{pmatrix}$, find the large-sample distribution of $\hat{\beta}_2$. Write your answer in terms of a, b, c and ν, where a, b, c are defined by

$$a = \sum \mu_i^2, \quad b = \sum z_i \mu_i^2, \quad c = \sum z_i^2 \mu_i^2.$$

1/I/6E **Mathematical Biology**

Consider a biological system in which concentrations $x(t)$ and $y(t)$ satisfy

$$\frac{dx}{dt} = f(y) - x \quad \text{and} \quad \frac{dy}{dt} = g(x) - y \ ,$$

where f and g are positive and monotonically decreasing functions of their arguments, so that x represses the synthesis of y and vice versa.

(a) Suppose the functions f and g are bounded. Sketch the phase plane and explain why there is always at least one steady state. Show that if there is a steady state with

$$\frac{\partial \ln f}{\partial \ln y} \, \frac{\partial \ln g}{\partial \ln x} > 1$$

then the system is multistable.

(b) If $f = \lambda/(1+y^m)$ and $g = \lambda/(1+x^n)$, where λ, m and n are positive constants, what values of m and n allow the system to display multistability for some value of λ?

Can $f = \lambda/y^m$ and $g = \lambda/x^n$ generate multistability? Explain your answer carefully.

2/I/6E **Mathematical Biology**

Consider a system with stochastic reaction events

$$x \xrightarrow{\lambda} x+1 \quad \text{and} \quad x \xrightarrow{\beta x^2} x-2 \ ,$$

where λ and β are rate constants.

(a) State or derive the exact differential equation satisfied by the average number of molecules $<x>$. Assuming that fluctuations are negligible, approximate the differential equation to obtain the steady-state value of $<x>$.

(b) Using this approximation, calculate the elasticity H, the average lifetime τ, and the average chemical event size $<r>$ (averaged over fluxes).

(c) State the stationary Fluctuation Dissipation Theorem for the normalised variance η. Hence show that

$$\eta = \frac{3}{4<x>} \ .$$

2/II/13E Mathematical Biology

Consider the reaction-diffusion system

$$\frac{\partial u}{\partial \tau} = \beta_u \left(\frac{u^2}{v} - u\right) + D_u \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial v}{\partial \tau} = \beta_v \left(u^2 - v\right) + D_v \frac{\partial^2 v}{\partial x^2}$$

for an activator u and inhibitor v, where β_u and β_v are degradation rate constants and D_u and D_v are diffusion rate constants.

(a) Find a suitably scaled time t and length s such that

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{u^2}{v} - u + \frac{\partial^2 u}{\partial s^2} \\ \frac{1}{Q}\frac{\partial v}{\partial t} &= u^2 - v + P\,\frac{\partial^2 v}{\partial s^2}\ , \end{aligned} \qquad (*)$$

and find expressions for P and Q.

(b) Show that the Jacobian matrix for small spatially homogenous deviations from a nonzero steady state of $(*)$ is

$$J = \begin{pmatrix} 1 & -1 \\ 2Q & -Q \end{pmatrix}$$

and find the values of Q for which the steady state is stable.
[*Hint: The eigenvalues of a* 2×2 *real matrix both have positive real parts iff the matrix has a positive trace and determinant.*]

(c) Derive linearised ordinary differential equations for the amplitudes $\hat{u}(t)$ and $\hat{v}(t)$ of small spatially inhomogeneous deviations from a steady state of $(*)$ that are proportional to $\cos(s/L)$, where L is a constant.

(d) Assuming that the system is stable to homogeneous perturbations, derive the condition for inhomogeneous *instability*. Interpret this condition in terms of how far activator and inhibitor molecules diffuse on average before they are degraded.

(e) Calculate the lengthscale L_{crit} of disturbances that are expected to be observed when the condition for inhomogeneous instability is just satisfied. What are the dominant mechanisms for stabilising disturbances on lengthscales (i) much less than and (ii) much greater than L_{crit}?

3/I/6E **Mathematical Biology**

Let x be the concentration of a binary master sequence of length L and let y be the total concentration of all mutant sequences. Master sequences try to self-replicate at a total rate ax, but each independent digit is only copied correctly with probability q. Mutant sequences self-replicate at a total rate by, where $a > b$, and the probability of mutation back to the master sequence is negligible.

(a) The evolution of x is given by

$$\frac{dx}{dt} = aq^L x \ .$$

Write down the corresponding equation for y and derive a differential equation for the master-to-mutant ratio $z = x/y$.

(b) What is the maximum length $L_{\max}$ for which there is a positive steady-state value of z? Is the positive steady state stable when it exists?

(c) Obtain a first-order approximation to $L_{\max}$ assuming that both $1 - q \ll 1$ and $s \ll 1$, where the selection coefficient s is defined by $b = a(1 - s)$.

3/II/13E **Mathematical Biology**

Protein synthesis by RNA can be represented by the stochastic system

$$\begin{aligned} x_1 &\xrightarrow{\lambda_1} x_1 + 1 \quad \text{and} \quad x_1 \xrightarrow{\beta_1 x_1} x_1 - 1 \\ x_2 &\xrightarrow{\lambda_2 x_1} x_2 + 1 \quad \text{and} \quad x_2 \xrightarrow{\beta_2 x_2} x_2 - 1 \end{aligned} \tag{1}$$

in which x_1 is an environmental variable corresponding to the number of RNA molecules per cell and x_2 is a system variable, with birth rate proportional to x_1, corresponding to the number of protein molecules.

(a) Use the normalized stationary Fluctuation–Dissipation Theorem (FDT) to calculate the (exact) normalized stationary variances $\eta_{11} = \sigma_1^2/{<}x_1{>}^2$ and $\eta_{22} = \sigma_2^2/{<}x_2{>}^2$ in terms of the averages ${<}x_1{>}$ and ${<}x_2{>}$.

(b) Separate η_{22} into an intrinsic and an extrinsic term by considering the limits when x_1 does not fluctuate (intrinsic), and when x_2 responds deterministically to changes in x_1 (extrinsic). Explain how the extrinsic term represents the magnitude of environmental fluctuations and time-averaging.

(c) Assume now that the birth rate of x_2 is changed from the "constitutive" mechanism $\lambda_2 x_1$ in (1) to a "negative feedback" mechanism $\lambda_2 x_1 f(x_2)$, where f is a monotonically decreasing function of x_2. Use the stationary FDT to approximate η_{22} in terms of $h = |\partial \ln f / \partial \ln x_2|$. Apply your answer to the case $f(x_2) = k/x_2$.

[*Hint: To reduce the algebra introduce the elasticity* $H_{22} = \partial \ln(R_2^-/R_2^+)/\partial \ln x_2$, *where* R_2^- *and* R_2^+ *are the death and birth rates of* x_2 *respectively.*]

(d) Explain the extrinsic term for the negative feedback system in terms of environmental fluctuations, time-averaging, and static susceptibility.

(e) Explain why the FDT is exact for the constitutive system but approximate for the feedback system. When, generally speaking, does the FDT approximation work well?

(f) Consider the following three experimental observations: (i) Large changes in λ_2 have no effect on η_{22}; (ii) When x_2 is perturbed by 1% from its stationary average, perturbations are corrected more rapidly in the feedback system than in the constitutive system; (iii) The feedback system displays lower values η_{22} than the constitutive system.

What does (i) imply about the relative importance of the noise terms? Can (ii) be directly explained by (iii), i.e., does rapid adjustment reduce noise? Justify your answers.

4/I/6E **Mathematical Biology**

The output of a linear perceptron is given by $y = \mathbf{w} \cdot \mathbf{x}$, where $\mathbf{w}$ is a vector of weights connecting a fluctuating input vector $\mathbf{x}$ to an output unit. The weights are given random initial values and are then updated according to a learning rule that has a time-constant τ much greater than the fluctuation timescale of the inputs.

(a) Find the behaviour of $|\mathbf{w}|$ for each of the following two rules

(i) $\tau \dfrac{d\mathbf{w}}{dt} = y\mathbf{x}$

(ii) $\tau \dfrac{d\mathbf{w}}{dt} = y\mathbf{x} - \alpha y^2 \mathbf{w}|\mathbf{w}|^2$, where α is a positive constant.

(b) Consider a third learning rule

(iii) $\tau \dfrac{d\mathbf{w}}{dt} = y\mathbf{x} - \mathbf{w}|\mathbf{w}|^2 \, .$

Show that in a steady state the vector of weights satisfies the eigenvalue equation

$$\mathbf{C}\mathbf{w} = \lambda \mathbf{w} \, ,$$

where the matrix $\mathbf{C}$ and eigenvalue λ should be identified.

(c) Comment briefly on the biological implications of the three rules.

1/I/7B **Dynamical Systems**

State Dulac's Criterion and the Poincaré–Bendixson Theorem regarding the existence of periodic solutions to the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Hence show that

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + y(\mu - 2x^2 - y^2)\end{aligned}$$

has no periodic solutions if $\mu < 0$ and at least one periodic solution if $\mu > 0$.

1/II/14B **Dynamical Systems**

Consider the equations

$$\begin{aligned}\dot{x} &= (a - x^2)(a^2 - y) \\ \dot{y} &= x - y\end{aligned}$$

as a function of the parameter a. Find the fixed points and plot their location in the (a, x) plane. Hence, or otherwise, deduce that there are bifurcations at $a = 0$ and $a = 1$.

Investigate the bifurcation at $a = 1$ by making the substitutions $u = x - 1$, $v = y - x$ and $\mu = a - 1$. Find the equation of the extended centre manifold to second order. Find the evolution equation on the centre manifold to second order, and determine the stability of its fixed points.

Show which branches of fixed points in the (a, x) plane are stable and which are unstable, and state, without calculation, the type of bifurcation at $a = 0$. Hence sketch the structure of the (x, y) phase plane very near the origin for $|a| \ll 1$ in the cases (i) $a < 0$ and (ii) $a > 0$.

The system is perturbed to $\dot{x} = (a - x^2)(a^2 - y) + \epsilon$, where $0 < \epsilon \ll 1$, with $\dot{y} = x - y$ still. Sketch the possible changes to the bifurcation diagram near $a = 0$ and $a = 1$. [*Calculation is not required.*]

2/I/7B Dynamical Systems

Define Lyapunov stability and quasi-asymptotic stability of a fixed point $\mathbf{x}_0$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

By considering a Lyapunov function of the form $V = g(x) + y^2$, show that the origin is an asymptotically stable fixed point of

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x^5 \,. \end{aligned}$$

[*Lyapunov's Second Theorem may be used without proof, provided you show that its conditions apply.*]

2/II/14B Dynamical Systems

Prove that if a continuous map F of an interval into itself has a periodic orbit of period three then it also has periodic orbits of least period n for all positive integers n.

Explain briefly why there must be at least two periodic orbits of least period 5.

[*You may assume without proof:*

(i) If U and V are non-empty closed bounded intervals such that $V \subseteq F(U)$ then there is a closed bounded interval $K \subseteq U$ such that $F(K) = V$.

(ii) The Intermediate Value Theorem.]

3/I/7B Dynamical Systems

Define the stable and unstable invariant subspaces of the linearisation of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ at a saddle point located at the origin in $\mathbb{R}^n$. How, according to the Stable Manifold Theorem, are the stable and unstable manifolds related to the invariant subspaces?

Calculate the stable and unstable manifolds, correct to cubic order, for the system

$$\begin{aligned} \dot{x} &= x + x^2 + 2xy + 3y^2 \\ \dot{y} &= -y + 3x^2 \,. \end{aligned}$$

4/I/7B **Dynamical Systems**

Find and classify the fixed points of the system

$$\begin{aligned}\dot{x} &= x(1-y)\\ \dot{y} &= -y + x^2\,.\end{aligned}$$

Sketch the phase plane.

What is the ω-limit for the point $(2,-1)$? Which points have $(0,0)$ as their ω-limit?

1/I/8A **Further Complex Methods**

Explain what is meant by the Papperitz symbol

$$P\left\{\begin{matrix} z_1 & z_2 & z_3 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}.$$

The hypergeometric function $F(a,b;c;z)$ is defined as the solution of the equation determined by the Papperitz symbol

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$$

that is analytic at $z=0$ and satisfies $F(a,b;c;0)=1$.

Show, explaining each step, that

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z).$$

2/I/8A **Further Complex Methods**

The Hankel representation of the gamma function is

$$\Gamma(z)=\frac{1}{2i\sin(\pi z)}\int_{-\infty}^{(0^+)} t^{z-1}e^t dt\ ,$$

where the path of integration is the Hankel contour.

Use this representation to find the residue of $\Gamma(z)$ at $z=-n$, where n is a non-negative integer.

Is there a pole at $z=n$, where n is a positive integer? Justify your answer carefully, working only from the above representation of $\Gamma(z)$.

3/I/8A **Further Complex Methods**

The functions f and g have Laplace transforms $\widehat{f}$ and $\widehat{g}$, and satisfy $f(t)=0=g(t)$ for $t<0$. The convolution h of f and g is defined by

$$h(u)=\int_0^u f(u-v)g(v)dv$$

and has Laplace transform $\widehat{h}$. Prove (the convolution theorem) that $\widehat{h}(p)=\widehat{f}(p)\widehat{g}(p)$.

Given that $\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds=\pi \quad (t>0)$, deduce the Laplace transform of the function $f(t)$, where

$$f(t)=\begin{cases} t^{-1/2}, & t>0 \\ 0, & t\leqslant 0. \end{cases}$$

3/II/14A **Further Complex Methods**

Show that the equation

$$zw''+2kw'+zw=0\,,$$

where k is constant, has solutions of the form

$$w(z)=\int_\gamma (t^2+1)^{k-1}e^{zt}dt$$

provided that the path γ is chosen so that $\left[(t^2+1)^k e^{zt}\right]_\gamma=0$.

(i) In the case $\operatorname{Re}k>0$, show that there is a choice of γ for which $w(0)=iB(k,\frac{1}{2})$.

(ii) In the case $k=n/2$, where n is any integer, show that γ can be a finite contour and that the corresponding solution satisfies $w(0)=0$ if $n\leqslant -1$.

4/I/8A Further Complex Methods

Write down necessary and sufficient conditions on the functions $p(z)$ and $q(z)$ for the point $z = 0$ to be (i) an ordinary point and (ii) a regular singular point of the equation

$$w'' + p(z)w' + q(z)w = 0. \qquad (*)$$

Show that the point $z = \infty$ is an ordinary point if and only if

$$p(z) = 2z^{-1} + z^{-2}P(z^{-1}), \qquad q(z) = z^{-4}Q(z^{-1}),$$

where P and Q are analytic in a neighbourhood of the origin.

Find the most general equation of the form $(*)$ that has a regular singular point at $z = 0$ but no other singular points.

4/II/14A Further Complex Methods

Two representations of the zeta function are

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1}dt \qquad \text{and} \qquad \zeta(z) = \sum_{1}^{\infty} n^{-z} \ ,$$

where, in the integral representation, the path is the Hankel contour and the principal branch of t^{z-1}, for which $|\arg z| < \pi$, is to be used. State the range of z for which each representation is valid.

Evaluate the integral

$$\int_{\gamma} \frac{t^{z-1}}{e^{-t}-1}dt,$$

where γ is a closed path consisting of the straight line $z = \pi i + x$, with $|x| < 2N\pi$, and the semicircle $|z - \pi i| = 2N\pi$, with $\mathrm{Im}\, z > \pi$, where N is a positive integer.

Making use of this result and assuming, when necessary, that the integral along the curved part of γ is negligible when N is large, derive the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z)$$

for $z \neq 1$.

1/I/9C Classical Dynamics

A particle of mass m_1 is constrained to move on a circle of radius r_1, centre $x = y = 0$ in a horizontal plane $z = 0$. A second particle of mass m_2 moves on a circle of radius r_2, centre $x = y = 0$ in a horizontal plane $z = c$. The two particles are connected by a spring whose potential energy is

$$V = \tfrac{1}{2}\omega^2 d^2,$$

where d is the distance between the particles. How many degrees of freedom are there? Identify suitable generalized coordinates and write down the Lagrangian of the system in terms of them.

1/II/15C Classical Dynamics

(i) The action for a system with generalized coordinates (q_a) is given by

$$S = \int_{t_1}^{t_2} L(q_a, \dot{q}_b)\, dt.$$

Derive Lagrange's equations from the principle of least action by considering all paths with fixed endpoints, $\delta q_a(t_1) = \delta q_a(t_2) = 0$.

(ii) A pendulum consists of a point mass m at the end of a light rod of length l. The pivot of the pendulum is attached to a mass M which is free to slide without friction along a horizontal rail. Choose as generalized coordinates the position x of the pivot and the angle θ that the pendulum makes with the vertical.

Write down the Lagrangian and derive the equations of motion.

Find the frequency of small oscillations around the stable equilibrium.

Now suppose that a force acts on the pivot causing it to travel with constant acceleration in the x-direction. Find the equilibrium angle θ of the pendulum.

2/I/9C **Classical Dynamics**

A rigid body has principal moments of inertia I_1, I_2 and I_3 and is moving under the action of no forces with angular velocity components $(\omega_1, \omega_2, \omega_3)$. Its motion is described by Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0\,. \end{aligned}$$

Are the components of the angular momentum to be evaluated in the body frame or the space frame?

Now suppose that an asymmetric body is moving with constant angular velocity $(\Omega, 0, 0)$. Show that this motion is stable if and only if I_1 is the largest or smallest principal moment.

3/I/9C **Classical Dynamics**

Define the Poisson bracket $\{f, g\}$ between two functions $f(q_a, p_a)$ and $g(q_a, p_a)$ on phase space. If $f(q_a, p_a)$ has no explicit time dependence, and there is a Hamiltonian H, show that Hamilton's equations imply

$$\frac{df}{dt} = \{f, H\}\,.$$

A particle with position vector $\mathbf{x}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Compute $\{p_a, L_b\}$ and $\{L_a, L_b\}$.

3/II/15C Classical Dynamics

(i) A point mass m with position q and momentum p undergoes one-dimensional periodic motion. Define the action variable I in terms of q and p. Prove that an orbit of energy E has period

$$T = 2\pi \frac{dI}{dE} .$$

(ii) Such a system has Hamiltonian

$$H(q,p) = \frac{p^2 + q^2}{\mu^2 - q^2} ,$$

where μ is a positive constant and $|q| < \mu$ during the motion. Sketch the orbits in phase space both for energies $E \gg 1$ and $E \ll 1$. Show that the action variable I is given in terms of the energy E by

$$I = \frac{\mu^2}{2} \frac{E}{\sqrt{E+1}} .$$

Hence show that for $E \gg 1$ the period of the orbit is $T \approx \frac{1}{2}\pi\mu^3/p_0$, where p_0 is the greatest value of the momentum during the orbit.

4/I/9C Classical Dynamics

Define a canonical transformation for a one-dimensional system with coordinates $(q,p) \to (Q,P)$. Show that if the transformation is canonical then $\{Q,P\} = 1$.

Find the values of constants α and β such that the following transformations are canonical:

(i) $Q = pq^\beta,\ P = \alpha q^{-1}$.

(ii) $Q = q^\alpha \cos(\beta p),\ P = q^\alpha \sin(\beta p)$.

1/I/10D **Cosmology**

(a) Around $t \approx 1\,\mathrm{s}$ after the big bang ($kT \approx 1\,\mathrm{MeV}$), neutrons and protons are kept in equilibrium by weak interactions such as

$$n + \nu_e \leftrightarrow p + e^- \,. \qquad (*)$$

Show that, in equilibrium, the neutron-to-proton ratio is given by

$$\frac{n_n}{n_p} \approx e^{-Q/kT} \,,$$

where $Q = (m_n - m_p)c^2 = 1.29\,\mathrm{MeV}$ corresponds to the mass difference between the neutron and the proton. Explain briefly why we can neglect the difference $\mu_n - \mu_p$ in the chemical potentials.

(b) The ratio of the weak interaction rate $\Gamma_W \propto T^5$ which maintains $(*)$ to the Hubble expansion rate $H \propto T^2$ is given by

$$\frac{\Gamma_W}{H} \approx \left(\frac{kT}{0.8\,\mathrm{MeV}}\right)^3 . \qquad (\dagger)$$

Explain why the neutron-to-proton ratio effectively "freezes out" once $kT < 0.8\,\mathrm{MeV}$, except for some slow neutron decay. Also explain why almost all neutrons are subsequently captured in ^4He; estimate the value of the relative mass density $Y_{^4He} = \rho_{^4He}/\rho_{\mathrm{B}}$ (with $\rho_{\mathrm{B}} = \rho_n + \rho_p$) given a final ratio $n_n/n_p \approx 1/8$.

(c) Suppose instead that the weak interaction rate were very much weaker than that described by equation $(\dagger)$. Describe the effect on the relative helium density $Y_{^4He}$. Briefly discuss the wider implications of this primordial helium-to-hydrogen ratio on stellar lifetimes and life on earth.

2/I/10D **Cosmology**

(a) A spherically symmetric star obeys the pressure-support equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \tag{$*$}$$

where $P(r)$ is the pressure at a distance r from the centre, $\rho(r)$ is the density, and the mass $m(r)$ is defined through the relation $dm/dr = 4\pi r^2\rho(r)$. Multiply $(*)$ by $4\pi r^3$ and integrate over the total volume V of the star to derive the virial theorem

$$\langle P\rangle V = -\tfrac{1}{3}E_{\rm grav}\,,$$

where $\langle P\rangle$ is the average pressure and $E_{\rm grav}$ is the total gravitational potential energy.

(b) Consider a white dwarf supported by electron Fermi degeneracy pressure $P \approx h^2 n^{5/3}/m_{\rm e}$, where $m_{\rm e}$ is the electron mass and n is the number density. Assume a uniform density $\rho(r) = m_{\rm p}n(r) \approx m_{\rm p}\langle n\rangle$, so the total mass of the star is given by $M = (4\pi/3)\langle n\rangle m_{\rm p}R^3$ where R is the star radius and $m_{\rm p}$ is the proton mass. Show that the total energy of the white dwarf can be written in the form

$$E_{\rm total} \;=\; E_{\rm kin} + E_{\rm grav} \;=\; \frac{\alpha}{R^2} \;-\; \frac{\beta}{R}\,,$$

where α, β are positive constants which you should determine. [*You may assume that for an ideal gas* $E_{\rm kin} = \frac{3}{2}\langle P\rangle V$.] Use this expression to explain briefly why a white dwarf is stable.

2/II/15D **Cosmology**

(a) Consider a homogeneous and isotropic universe with scale factor $a(t)$ and filled with mass density $\rho(t)$. Show how the conservation of kinetic energy plus gravitational potential energy for a test particle on the edge of a spherical region in this universe can be used to derive the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho\,, \tag{$*$}$$

where k is a constant. State clearly any assumptions you have made.

(b) Now suppose that the universe was filled throughout its history with radiation with equation of state $P = \rho c^2/3$. Using the fluid conservation equation and the definition of the relative density Ω, show that the density of this radiation can be expressed as

$$\rho \;=\; \frac{3H_0^2}{8\pi G}\frac{\Omega_0}{a^4}\,,$$

where H_0 is the Hubble parameter today and Ω_0 is the relative density today ($t = t_0$) and $a_0 \equiv a(t_0) = 1$ is assumed. Show also that $kc^2 = H_0^2(\Omega_0 - 1)$ and hence rewrite the Friedmann equation ($*$) as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_0\left(\frac{1}{a^4} - \frac{\beta}{a^2}\right)\,, \tag{$\dagger$}$$

where $\beta \equiv (\Omega_0 - 1)/\Omega_0$.

(c) Now consider a closed model with $k > 0$ (or $\Omega > 1$). Rewrite ($\dagger$) using the new time variable τ defined by

$$\frac{dt}{d\tau} = a\,.$$

Hence, or otherwise, solve ($\dagger$) to find the parametric solution

$$a(\tau) = \frac{1}{\sqrt{\beta}}(\sin\alpha\tau)\,, \qquad t(\tau) = \frac{1}{\alpha\sqrt{\beta}}(1 - \cos\alpha\tau)\,,$$

where $\alpha \equiv H_0\sqrt{(\Omega_0 - 1)}$. [*Recall that* $\int dx/\sqrt{1-x^2} = \sin^{-1}x$.]

Using the solution for $a(\tau)$, find the value of the new time variable $\tau = \tau_0$ today and hence deduce that the age of the universe in this model is

$$t_0 = H_0^{-1}\frac{\sqrt{\Omega_0} - 1}{\Omega_0 - 1}\,.$$

3/I/10D **Cosmology**

(a) Define and discuss the concept of the *cosmological horizon* and the *Hubble radius* for a homogeneous isotropic universe. Illustrate your discussion with the specific examples of the Einstein–de Sitter universe ($a \propto t^{2/3}$ for $t > 0$) and a de Sitter universe ($a \propto e^{Ht}$ with H constant, $t > -\infty$).

(b) Explain the *horizon problem* for a decelerating universe in which $a(t) \propto t^{\alpha}$ with $\alpha < 1$. How can inflation cure the horizon problem?

(c) Consider a Tolman (radiation-filled) universe ($a(t) \propto t^{1/2}$) beginning at $t_{\mathrm{r}} \sim 10^{-35}$s and lasting until today at $t_0 \approx 10^{17}$s. Estimate the horizon size today $d_H(t_0)$ and project this lengthscale backwards in time to show that it had a physical size of about 1 metre at $t \approx t_{\mathrm{r}}$.

Prior to $t \approx t_{\mathrm{r}}$, assume an inflationary (de Sitter) epoch with constant Hubble parameter H given by its value at $t \approx t_{\mathrm{r}}$ for the Tolman universe. How much expansion during inflation is required for the observable universe today to have begun inside one Hubble radius?

4/I/10D **Cosmology**

The linearised equation for the growth of a density fluctuation δ_k in a homogeneous and isotropic universe is

$$\frac{d^2\delta_k}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta_k}{dt} - \left(4\pi G\rho_{\mathrm{m}} - \frac{v_s^2 k^2}{a^2}\right)\delta_k = 0\,, \qquad (*)$$

where ρ_{m} is the non-relativistic matter density, k is the comoving wavenumber and v_s is the sound speed ($v_s^2 \equiv dP/d\rho$).

(a) Define the Jeans length λ_{J} and discuss its significance for perturbation growth.

(b) Consider an Einstein–de Sitter universe with $a(t) = (t/t_0)^{2/3}$ filled with pressure-free matter ($P = 0$). Show that the perturbation equation $(*)$ can be re-expressed as

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0\,.$$

By seeking power law solutions, find the growing and decaying modes of this equation.

(c) Qualitatively describe the evolution of non-relativistic matter perturbations ($k > aH$) in the radiation era, $a(t) \propto t^{1/2}$, when $\rho_{\mathrm{r}} \gg \rho_{\mathrm{m}}$. What feature in the power spectrum is associated with the matter–radiation transition?

4/II/15D **Cosmology**

For an ideal gas of *bosons*, the average occupation number can be expressed as

$$\bar{n}_k = \frac{g_k}{e^{(E_k-\mu)/kT}-1}\,, \qquad (*)$$

where g_k has been included to account for the degeneracy of the energy level E_k. In the approximation in which a discrete set of energies E_k is replaced with a continuous set with momentum p, the density of one-particle states with momentum in the range p to $p+dp$ is $g(p)dp$. Explain briefly why

$$g(p) \;\propto\; p^2 V\,,$$

where V is the volume of the gas. Using this formula with equation $(*)$, obtain an expression for the total energy density $\epsilon = E/V$ of an ultra-relativistic gas of bosons at zero chemical potential as an integral over p. Hence show that

$$\epsilon \;\propto\; T^\alpha\,,$$

where α is a number you should find. Why does this formula apply to photons?

Prior to a time $t \sim 100,000$ years, the universe was filled with a gas of photons and non-relativistic free electrons and protons. Subsequently, at around $t \sim 400,000$ years, the protons and electrons began combining to form neutral hydrogen,

$$p + e^- \leftrightarrow H + \gamma\,.$$

Deduce Saha's equation for this recombination process stating clearly the steps required:

$$\frac{n_{\rm e}^2}{n_{\rm H}} = \left(\frac{2\pi m_{\rm e} kT}{h^2}\right)^{3/2} \exp(-I/kT)\,,$$

where I is the ionization energy of hydrogen. [*Note that the equilibrium number density of a non-relativistic species* ($kT \ll mc^2$) *is given by* $n = g_s\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\exp\left[(\mu - mc^2)/kT\right]$, *while the photon number density is* $n_\gamma = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3$, where $\zeta(3) \approx 1.20....$]

Consider now the fractional ionization $X_{\rm e} = n_{\rm e}/n_{\rm B}$, where $n_B = n_{\rm p} + n_{\rm H} = \eta n_\gamma$ is the baryon number of the universe and η is the baryon-to-photon ratio. Find an expression for the ratio

$$(1 - X_{\rm e})/X_{\rm e}^2$$

in terms only of kT and constants such as η and I. One might expect neutral hydrogen to form at a temperature given by $kT \approx I \approx 13\,\text{eV}$, but instead in our universe it forms at the much lower temperature $kT \approx 0.3\,\text{eV}$. Briefly explain why.

1/II/16F Logic and Set Theory

State and prove Zorn's Lemma. [*You may assume Hartogs' Lemma.*] Where in your argument have you made use of the Axiom of Choice?

Show that $\mathbb{R}$, considered as a rational vector space, has a basis.

Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as rational vector spaces.

2/II/16F Logic and Set Theory

Give the inductive and the synthetic definitions of ordinal addition, and prove that they are equivalent. Give an example to show that ordinal addition is not commutative.

Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(ii) If α and β are limit ordinals then $\alpha + \beta = \beta + \alpha$.

(iii) If $\alpha + \beta = \omega_1$ then $\alpha = 0$ or $\alpha = \omega_1$.

(iv) If $\alpha + \beta = \omega_1$ then $\beta = 0$ or $\beta = \omega_1$.

3/II/16F Logic and Set Theory

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent (in the presence of the other axioms of ZF). [*You may assume the existence of transitive closures.*]

Explain briefly how the Principle of $\in$-Induction implies that every set is a member of some V_α.

For each natural number n, find the cardinality of V_n. For which ordinals α is the cardinality of V_α equal to that of the reals?

4/II/16F Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic. [*You do* **not** *need to give definitions of the various terms involved. You may assume that the set of primitive propositions is countable. You may also assume the Deduction Theorem, provided that you state it clearly.*]

Where in your argument have you used the third axiom, namely $(\neg\neg p) \Rightarrow p$?

State the Compactness Theorem, and deduce it from the Completeness Theorem.

1/II/17F Graph Theory

Show that an acyclic graph has a vertex of degree at most one. Prove that a tree (that is, a connected acyclic graph) of order n has size $n-1$, and deduce that every connected graph of order n and size $n-1$ is a tree.

Let T be a tree of order t. Show that if G is a graph with $\delta(G) \geqslant t-1$ then T is a subgraph of G, but that this need not happen if $\delta(G) \geqslant t-2$.

2/II/17F Graph Theory

Brooks' Theorem states that if G is a connected graph then $\chi(G) \leqslant \Delta(G)$ unless G is complete or is an odd cycle. Prove the theorem for 3-connected graphs G.

Let G be a graph, and let $d_1 + d_2 = \Delta(G) - 1$. By considering a partition V_1, V_2 of $V(G)$ that minimizes the quantity $d_2 e(G[V_1]) + d_1 e(G[V_2])$, show that there is a partition with $\Delta(G[V_i]) \leqslant d_i$, $i = 1, 2$.

By taking $d_1 = 3$, show that if a graph G contains no K_4 then $\chi(G) \leqslant \frac{3}{4}\Delta(G) + \frac{3}{2}$.

3/II/17F Graph Theory

Let X and Y be disjoint sets of $n \geqslant 6$ vertices each. Let G be a bipartite graph formed by adding edges between X and Y randomly and independently with probability $p = 1/100$. Let $e(U, V)$ be the number of edges of G between the subsets $U \subset X$ and $V \subset Y$. Let $k = \lceil n^{1/2} \rceil$. Consider three events $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, as follows.

$\mathcal{A}$: there exist $U \subset X$, $V \subset Y$ with $|U| = |V| = k$ and $e(U, V) = 0$

$\mathcal{B}$: there exist $x \in X$, $W \subset Y$ with $|W| = n - k$ and $e(\{x\}, W) = 0$

$\mathcal{C}$: there exist $Z \subset X$, $y \in Y$ with $|Z| = n - k$ and $e(Z, \{y\}) = 0$.

Show that $\Pr(\mathcal{A}) \leqslant n^{2k}(1-p)^{k^2}$ and $\Pr(\mathcal{B} \cup \mathcal{C}) \leqslant 2n^{k+1}(1-p)^{n-k}$. Hence show that $\Pr(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) < 3n^{2k}(1-p)^{n/2}$ and so show that, almost surely, none of $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ occur. Deduce that, almost surely, G has a matching from X to Y.

4/II/17F Graph Theory

Write an essay on extremal graph theory. Your essay should include the proof of at least one extremal theorem. You should state the Erdős–Stone theorem, as well as describing its proof and showing how it can be applied.

1/II/18G Galois Theory

Let L/K be a field extension. State what it means for an element $x \in L$ to be *algebraic* over K. Show that x is algebraic over K if and only if the field $K(x)$ is finite dimensional as a vector space over K.

State what it means for a field extension L/K to be *algebraic*. Show that, if M/L is algebraic and L/K is algebraic, then M/K is algebraic.

2/II/18G Galois Theory

Let K be a field of characteristic 0 containing all roots of unity.

(i) Let L be the splitting field of the polynomial $X^n - a$ where $a \in K$. Show that the Galois group of L/K is cyclic.

(ii) Suppose that M/K is a cyclic extension of degree m over K. Let g be a generator of the Galois group and $\zeta \in K$ a primitive m-th root of 1. By considering the resolvent

$$R(w) = \sum_{i=0}^{m-1} \frac{g^i(w)}{\zeta^i}$$

of elements $w \in M$, show that M is the splitting field of a polynomial $X^m - a$ for some $a \in K$.

3/II/18G Galois Theory

Find the Galois group of the polynomial

$$x^4 + x + 1$$

over $\mathbb{F}_2$ and $\mathbb{F}_3$. Hence or otherwise determine the Galois group over $\mathbb{Q}$.

[*Standard general results from Galois theory may be assumed.*]

4/II/18G Galois Theory

(i) Let K be the splitting field of the polynomial

$$x^4 - 4x^2 - 1$$

over $\mathbb{Q}$. Show that $[K : \mathbb{Q}] = 8$, and hence show that the Galois group of $K/\mathbb{Q}$ is the dihedral group of order 8.

(ii) Let L be the splitting field of the polynomial

$$x^4 - 4x^2 + 1$$

over $\mathbb{Q}$. Show that $[L : \mathbb{Q}] = 4$. Show that the Galois group of $L/\mathbb{Q}$ is $C_2 \times C_2$.

1/II/19G **Representation Theory**

Let the finite group G act on finite sets X and Y, and denote by $\mathbb{C}[X]$, $\mathbb{C}[Y]$ the associated permutation representations on the spaces of complex functions on X and Y. Call their characters χ_X and χ_Y.

(i) Show that the inner product $\langle\chi_X|\chi_Y\rangle$ is the number of orbits for the diagonal action of G on $X \times Y$.

(ii) Assume that $|X| > 1$, and let $S \subset \mathbb{C}[X]$ be the subspace of those functions whose values sum to zero. By considering $\|\chi_X\|^2$, show that S is irreducible if and only if the G-action on X is *doubly transitive*: this means that for any two pairs (x_1, x_2) and (x'_1, x'_2) of points in X with $x_1 \neq x_2$ and $x'_1 \neq x'_2$, there exists some $g \in G$ with $gx_1 = x'_1$ and $gx_2 = x'_2$.

(iii) Let now $G = S_n$ acting on the set $X = \{1, 2, \ldots, n\}$. Call Y the set of 2-element subsets of X, with the natural action of S_n. If $n \geqslant 4$, show that $\mathbb{C}[Y]$ decomposes under S_n into three irreducible representations, one of which is the trivial representation and another of which is S. What happens when $n = 3$?

[*Hint: Consider* $\langle 1|\chi_Y\rangle$, $\langle\chi_X|\chi_Y\rangle$ *and* $\|\chi_Y\|^2$.]

2/II/19G **Representation Theory**

Let G be a finite group and $\{\chi_i\}$ the set of its irreducible characters. Also choose representatives g_j for the conjugacy classes, and denote by $Z(g_j)$ their centralisers.

(i) State the orthogonality and completeness relations for the χ_k.

(ii) Using Part (i), or otherwise, show that

$$\sum\nolimits_i \overline{\chi_i(g_j)} \cdot \chi_i(g_k) = \delta_{jk} \cdot |Z(g_j)|\,.$$

(iii) Let A be the matrix with $A_{ij} = \chi_i(g_j)$. Prove that

$$|\det A|^2 = \prod\nolimits_j |Z(g_j)|.$$

(iv) Show that $\det A$ is either real or purely imaginary, explaining when each situation occurs.

[*Hint for (iv): Consider the effect of complex conjugation on the rows of the matrix A.*]

3/II/19G Representation Theory

Let G be the group with 21 elements generated by a and b, subject to the relations $a^7 = b^3 = 1$ and $ba = a^2 b$.

(i) Find the conjugacy classes of G.

(ii) Find three non-isomorphic one-dimensional representations of G.

(iii) For a subgroup H of a finite group K, write down (without proof) the formula for the character of the K-representation induced from a representation of H.

(iv) By applying Part (iii) to the case when H is the subgroup $\langle a \rangle$ of $K = G$, find the remaining irreducible characters of G.

4/II/19G Representation Theory

(i) State and prove the Weyl integration formula for $SU(2)$.

(ii) Determine the characters of the symmetric powers of the standard 2-dimensional representation of $SU(2)$ and prove that they are irreducible.

[*Any general theorems from the course may be used.*]

1/II/20G Number Fields

Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ where p is a prime with $p \equiv 3 \pmod 4$. By computing the relative traces $\mathrm{Tr}_{K/k}(\theta)$ where k runs through the three quadratic subfields of K, show that the algebraic integers θ in K have the form

$$\theta = \tfrac{1}{2}(a + b\sqrt{p}) + \tfrac{1}{2}(c + d\sqrt{p})\sqrt{2}\,,$$

where a, b, c, d are rational integers. By further computing the relative norm $\mathrm{N}_{K/k}(\theta)$ where $k = \mathbb{Q}(\sqrt{2})$, show that 4 divides

$$a^2 + pb^2 - 2\big(c^2 + pd^2\big) \quad \text{and} \quad 2\big(ab - 2cd\big).$$

Deduce that a and b are even and $c \equiv d \pmod 2$. Hence verify that an integral basis for K is

$$1, \quad \sqrt{2}, \quad \sqrt{p}, \quad \tfrac{1}{2}\big(1 + \sqrt{p}\big)\sqrt{2}.$$

2/II/20G Number Fields

Show that $\varepsilon = (3 + \sqrt{7})/(3 - \sqrt{7})$ is a unit in $k = \mathbb{Q}(\sqrt{7})$. Show further that 2 is the square of the principal ideal in k generated by $3 + \sqrt{7}$.

Assuming that the Minkowski constant for k is $\frac{1}{2}$, deduce that k has class number 1.

Assuming further that ε is the fundamental unit in k, show that the complete solution in integers x, y of the equation $x^2 - 7y^2 = 2$ is given by

$$x + \sqrt{7}y = \pm\varepsilon^n(3 + \sqrt{7}) \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Calculate the particular solution in positive integers x, y when $n = 1$.

4/II/20G Number Fields

State Dedekind's theorem on the factorisation of rational primes into prime ideals.

A rational prime is said to ramify totally in a field with degree n if it is the n-th power of a prime ideal in the field. Show that, in the quadratic field $\mathbb{Q}(\sqrt{d})$ with d a square-free integer, a rational prime ramifies totally if and only if it divides the discriminant of the field.

Verify that the same holds in the cyclotomic field $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/q}$ with q an odd prime, and also in the cubic field $\mathbb{Q}(\sqrt[3]{2})$.

[*The cases* $d \equiv 2, 3 \pmod 4$ *and* $d \equiv 1 \pmod 4$ *for the quadratic field should be carefully distinguished. It can be assumed that* $\mathbb{Q}(\zeta)$ *has a basis* $1, \zeta, \ldots, \zeta^{q-2}$ *and discriminant* $(-1)^{(q-1)/2}q^{q-1}$, *and that* $\mathbb{Q}(\sqrt[3]{2})$ *has a basis* $1, \sqrt[3]{2}, (\sqrt[3]{2})^2$ *and discriminant* -108.]

1/II/21H Algebraic Topology

(i) Show that if $E \to T$ is a covering map for the torus $T = S^1 \times S^1$, then E is homeomorphic to one of the following: the plane $\mathbb{R}^2$, the cylinder $\mathbb{R} \times S^1$, or the torus T.

(ii) Show that any continuous map from a sphere S^n $(n \geqslant 2)$ to the torus T is homotopic to a constant map.

[*General theorems from the course may be used without proof, provided that they are clearly stated.*]

2/II/21H Algebraic Topology

State the Van Kampen Theorem. Use this theorem and the fact that $\pi_1 S^1 = \mathbb{Z}$ to compute the fundamental groups of the torus $T = S^1 \times S^1$, the punctured torus $T \setminus \{p\}$, for some point $p \in T$, and the connected sum $T \# T$ of two copies of T.

3/II/20H Algebraic Topology

Let X be a space that is triangulable as a simplicial complex with no n-simplices. Show that any continuous map from X to S^n is homotopic to a constant map.

[*General theorems from the course may be used without proof, provided they are clearly stated.*]

4/II/21H Algebraic Topology

Let X be a simplicial complex. Suppose $X = B \cup C$ for subcomplexes B and C, and let $A = B \cap C$. Show that the inclusion of A in B induces an isomorphism $H_* A \to H_* B$ if and only if the inclusion of C in X induces an isomorphism $H_* C \to H_* X$.

1/II/22F **Linear Analysis**

Let K be a compact Hausdorff space, and let $C(K)$ denote the Banach space of continuous, complex-valued functions on K, with the supremum norm. Define what it means for a set $S \subset C(K)$ to be *totally bounded*, *uniformly bounded*, and *equicontinuous*.

Show that S is totally bounded if and only if it is both uniformly bounded and equicontinuous.

Give, with justification, an example of a Banach space X and a subset $S \subset X$ such that S is bounded but not totally bounded.

2/II/22F **Linear Analysis**

Let X and Y be Banach spaces. Define what it means for a linear operator $T : X \to Y$ to be *compact*. For a linear operator $T : X \to X$, define the *spectrum*, *point spectrum*, and *resolvent set* of T.

Now let H be a complex Hilbert space. Define what it means for a linear operator $T : H \to H$ to be *self-adjoint*. Suppose $e_1, e_2, \ldots$ is an orthonormal basis for H. Define a linear operator $T : H \to H$ by setting $Te_i = \frac{1}{i} e_i$. Is T compact? Is T self-adjoint? Justify your answers. Describe, with proof, the spectrum, point spectrum, and resolvent set of T.

3/II/21F **Linear Analysis**

Let X be a normed vector space. Define the dual X^* of X. Define the normed vector spaces $l^s = l^s(\mathbb{C})$ for all $1 \leqslant s \leqslant \infty$. [*You are* **not** *required to prove that the norms you have given are indeed norms.*]

Now let $1 < p, q < \infty$ be such that $p^{-1} + q^{-1} = 1$. Show that $(l^q)^*$ is isometrically isomorphic to l^p as a normed vector space. [*You may assume any standard inequalities.*]

Show by a similar argument that $(l^1)^*$ is isomorphic to l^∞. Does your argument also show that $(l^\infty)^*$ is isomorphic to l^1? If not, where does it fail?

4/II/22F **Linear Analysis**

Let X and Y be normed vector spaces. Show that a linear map $T : X \to Y$ is continuous if and only if it is bounded.

Now let X, Y, Z be Banach spaces. We say that a map $F : X \times Y \to Z$ is *bilinear* if

$$F(\alpha x + \beta y, z) = \alpha F(x, z) + \beta F(y, z), \text{ for all scalars } \alpha, \beta \text{ and } x, y \in X, z \in Y$$

$$F(x, \alpha y + \beta z) = \alpha F(x, y) + \beta F(x, z), \text{ for all scalars } \alpha, \beta \text{ and } x \in X, y, z \in Y\,.$$

Suppose that F is bilinear and is continuous in each variable separately. Show that there exists a constant $M \geqslant 0$ such that

$$||F(x, y)|| \leqslant M||x||\;||y||$$

for all $x \in X$, $y \in Y$.

[*Hint: For each fixed* $x \in X$, *consider the map* $y \mapsto F(x, y)$ *from* Y *to* Z.]

1/II/23H **Riemann Surfaces**

Let Λ be a lattice in $\mathbb{C}$ generated by 1 and τ, where τ is a fixed complex number with $\mathrm{Im}\tau > 0$. The Weierstrass $\wp$-function is defined as a Λ-periodic meromorphic function such that

(1) the only poles of $\wp$ are at points of Λ, and

(2) there exist positive constants ε and M such that for all $|z| < \varepsilon$, we have

$$|\wp(z) - 1/z^2| < M|z|.$$

Show that $\wp$ is uniquely determined by the above properties and that $\wp(-z) = \wp(z)$. By considering the valency of $\wp$ at $z = 1/2$, show that $\wp''(1/2) \neq 0$.

Show that $\wp$ satisfies the differential equation

$$\wp''(z) = 6\wp^2(z) + A,$$

for some complex constant A.

[*Standard theorems about doubly-periodic meromorphic functions may be used without proof provided they are accurately stated, but any properties of the $\wp$-function that you use must be deduced from first principles.*]

2/II/23H **Riemann Surfaces**

Define the terms *function element* and *complete analytic function*.

Let (f, D) be a function element such that $f(z)^n = p(z)$, for some integer $n \geqslant 2$, where $p(z)$ is a complex polynomial with no multiple roots. Let F be the complete analytic function containing (f, D). Show that every function element $(\tilde{f}, \tilde{D})$ in F satisfies $\tilde{f}(z)^n = p(z)$.

Describe how the non-singular complex algebraic curve

$$C = \{(z, w) \in \mathbb{C}^2 \mid w^n - p(z) = 0\}$$

can be made into a Riemann surface such that the first and second projections $\mathbb{C}^2 \to \mathbb{C}$ define, by restriction, holomorphic maps $f_1, f_2 : C \to \mathbb{C}$.

Explain briefly the relation between C and the Riemann surface $S(F)$ for the complete analytic function F given earlier.

[*You do not need to prove the Inverse Function Theorem, provided that you state it accurately.*]

3/II/22H Riemann Surfaces

Explain what is meant by a *meromorphic differential* on a compact connected Riemann surface S. Show that if f is a meromorphic function on S then df defines a meromorphic differential on S. Show also that if η and ω are two meromorphic differentials on S which are not identically zero then $\eta = h\omega$ for some meromorphic function h. Show that zeros and poles of a meromorphic differential are well-defined and explain, without proof, how to obtain the genus of S by counting zeros and poles of ω.

Let $V_0 \subset \mathbb{C}^2$ be the affine curve with equation $u^2 = v^2 + 1$ and let $V \subset \mathbb{P}^2$ be the corresponding projective curve. Show that V is non-singular with two points at infinity, and that dv extends to a meromorphic differential on V.

[*You may assume without proof that that the map*

$$(u, v) = \left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right), \qquad t \in \mathbb{C} \setminus \{-1, 1\},$$

is onto $V_0 \setminus \{(1,0)\}$ *and extends to a biholomorphic map from* $\mathbb{P}^1$ *onto* V.]

4/II/23H Riemann Surfaces

Define what is meant by the *degree* of a non-constant holomorphic map between compact connected Riemann surfaces, and state the Riemann–Hurwitz formula.

Let $E_\Lambda = \mathbb{C}/\Lambda$ be an elliptic curve defined by some lattice Λ. Show that the map

$$\psi : z + \Lambda \in E_\Lambda \to -z + \Lambda \in E_\Lambda$$

is biholomorphic, and that there are four points in E_Λ fixed by ψ.

Let $S = E_\Lambda / \sim$ be the quotient surface (the topological surface obtained by identifying $z + \Lambda$ and $\psi(z + \Lambda)$, for each z) and let $\pi : E_\Lambda \to S$ be the corresponding projection map. Denote by $E_\Lambda^0 \subset E_\Lambda$ the complement of the four points fixed by ψ, and let $S^0 = \pi(E_\Lambda^0)$. Describe briefly a family of charts making S^0 into a Riemann surface, so that $\pi : E_\Lambda^0 \to S^0$ is a holomorphic map.

Now assume that the complex structure of S^0 extends to S, so that S is a Riemann surface, and that the map π is in fact holomorphic on all of E_Λ. Calculate the genus of S.

1/II/24H Differential Geometry

Let $f : X \to Y$ be a smooth map between manifolds without boundary.

(i) Define what is meant by a *critical point*, *critical value* and *regular value* of f.

(ii) Show that if y is a regular value of f and $\dim X \geqslant \dim Y$, then the set $f^{-1}(y)$ is a submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$.

[*You may assume the inverse function theorem.*]

(iii) Let $SL(n, \mathbb{R})$ be the group of all $n \times n$ real matrices with determinant 1. Prove that $SL(n, \mathbb{R})$ is a submanifold of the set of all $n \times n$ real matrices. Find the tangent space to $SL(n, \mathbb{R})$ at the identity matrix.

2/II/24H Differential Geometry

State the isoperimetric inequality in the plane.

Let $S \subset \mathbb{R}^3$ be a surface. Let $p \in S$ and let $S_r(p)$ be a geodesic circle of centre p and radius r (r small). Let L be the length of $S_r(p)$ and A be the area of the region bounded by $S_r(p)$. Prove that

$$4\pi A - L^2 = \pi^2 r^4 K(p) + \varepsilon(r),$$

where $K(p)$ is the Gaussian curvature of S at p and

$$\lim_{r \to 0} \frac{\varepsilon(r)}{r^4} = 0.$$

When $K(p) > 0$ and r is small, compare this briefly with the isoperimetric inequality in the plane.

3/II/23H Differential Geometry

(i) Define geodesic curvature and state the Gauss–Bonnet theorem.

(ii) Let $\alpha : I \to \mathbb{R}^3$ be a closed regular curve parametrized by arc-length, and assume that α has non-zero curvature everywhere. Let $n : I \to S^2 \subset \mathbb{R}^3$ be the curve given by the normal vector $n(s)$ to $\alpha(s)$. Let $\bar{s}$ be the arc-length of the curve n on S^2. Show that the geodesic curvature k_g of n is given by

$$k_g = -\frac{d}{ds} \tan^{-1}(\tau/k) \, \frac{ds}{d\bar{s}} \,,$$

where k and τ are the curvature and torsion of α.

(iii) Suppose now that $n(s)$ is a simple curve (i.e. it has no self-intersections). Show that $n(I)$ divides S^2 into two regions of equal area.

4/II/24H **Differential Geometry**

(i) Define what is meant by an *isothermal parametrization*. Let $\phi : U \to \mathbb{R}^3$ be an isothermal parametrization. Prove that

$$\phi_{uu} + \phi_{vv} = 2\,\lambda^2\,\mathbf{H},$$

where $\mathbf{H}$ is the mean curvature vector and $\lambda^2 = \langle \phi_u, \phi_u \rangle$.

Define what it means for ϕ to be *minimal*, and deduce that ϕ is minimal if and only if $\Delta\phi = 0$.

[*You may assume that the mean curvature H can be written as*

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}.\,]$$

(ii) Write $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$. Consider the complex valued functions

$$\varphi_1 = x_u - i x_v, \quad \varphi_2 = y_u - i y_v, \quad \varphi_3 = z_u - i z_v.$$

Show that ϕ is isothermal if and only if $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0$.

Suppose now that ϕ is isothermal. Prove that ϕ is minimal if and only if φ_1, φ_2 and φ_3 are holomorphic functions.

(iii) Consider the immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\phi(u,v) = (u - u^3/3 + uv^2,\ -v + v^3/3 - u^2 v,\ u^2 - v^2).$$

Find φ_1, φ_2 and φ_3. Show that ϕ is an isothermal parametrization of a minimal surface.

1/II/25J Probability and Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $\mathcal{G} \subseteq \mathcal{F}$, what is meant by saying that $\mathcal{G}$ is a π-system? State the 'uniqueness of extension' theorem for measures on $\sigma(\mathcal{G})$ having given values on $\mathcal{G}$.

For $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$, we call $\mathcal{G}$, $\mathcal{H}$ independent if

$$\mathbb{P}(G \cap H) = \mathbb{P}(G)\mathbb{P}(H) \quad \text{for all} \quad G \in \mathcal{G},\ H \in \mathcal{H}.$$

If $\mathcal{G}$ and $\mathcal{H}$ are independent π-systems, show that $\sigma(\mathcal{G})$ and $\sigma(\mathcal{H})$ are independent.

Let $Y_1, Y_2, \ldots, Y_m,\ Z_1, Z_2, \ldots, Z_n$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the σ-fields $\sigma(Y) = \sigma(Y_1, Y_2, \ldots, Y_m)$ and $\sigma(Z) = \sigma(Z_1, Z_2, \ldots, Z_n)$ are independent.

2/II/25J Probability and Measure

Let $\mathcal{R}$ be a family of random variables on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. What is meant by saying that $\mathcal{R}$ is uniformly integrable? Explain the use of uniform integrability in the study of convergence in probability and in L^1. [*Clear definitions should be given of any terms used, but proofs may be omitted.*]

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be uniformly integrable families of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family $\mathcal{R}$ given by

$$\mathcal{R} = \{X + Y : X \in \mathcal{R}_1,\ Y \in \mathcal{R}_2\}$$

is uniformly integrable.

3/II/24J Probability and Measure

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For a measurable function $f : \Omega \to \mathbb{R}$, and $p \in [1, \infty)$, let $||f||_p = [\mu(|f|^p)]^{1/p}$. Let L^p be the space of all such f with $||f||_p < \infty$. Explain what is meant by each of the following statements:

(a) A sequence of functions $(f_n : n \geqslant 1)$ is Cauchy in L^p.

(b) L^p is complete.

Show that L^p is complete for $p \in [1, \infty)$.

Take $\Omega = (1, \infty)$, $\mathcal{F}$ the Borel σ-field of Ω, and μ the Lebesgue measure on $(\Omega, \mathcal{F})$. For $p = 1, 2$, determine which if any of the following sequences of functions are Cauchy in L^p:

(i) $f_n(x) = x^{-1} 1_{(1,n)}(x)$,

(ii) $g_n(x) = x^{-2} 1_{(1,n)}(x)$,

where 1_A denotes the indicator function of the set A.

4/II/25J Probability and Measure

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel-measurable. State Fubini's theorem for the double integral

$$\int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} f(x, y)\, dx\, dy\ .$$

Let $0 < a < b$. Show that the function

$$f(x, y) = \begin{cases} e^{-xy} & \text{if } \ x \in (0, \infty),\ \ y \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

is measurable and integrable on $\mathbb{R}^2$.

Evaluate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx$$

by Fubini's theorem or otherwise.

1/II/26I Applied Probability

A cell has been placed in a biological solution at time $t = 0$. After an exponential time of rate μ, it is divided, producing k cells with probability p_k, $k = 0, 1, \ldots$, with the mean value $\rho = \sum_{k=1}^{\infty} k p_k$ ($k = 0$ means that the cell dies). The same mechanism is applied to each of the living cells, independently.

(a) Let M_t be the number of living cells in the solution by time $t > 0$. Prove that $\mathbb{E} M_t = \exp\left[t\mu(\rho - 1)\right]$. [*You may use without proof, if you wish, the fact that, if a positive function $a(t)$ satisfies $a(t+s) = a(t)a(s)$ for $t, s \geqslant 0$ and is differentiable at zero, then $a(t) = e^{\alpha t}$, $t \geqslant 0$, for some α.*]

Let $\phi_t(s) = \mathbb{E}\ s^{M_t}$ be the probability generating function of M_t. Prove that it satisfies the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(s) = \mu\left(-\phi_t(s) + \sum_{k=0}^{\infty} p_k \left[\phi_t(s)\right]^k\right), \quad \text{with} \quad \phi_0(s) = s.$$

(b) Now consider the case where each cell is divided in two cells ($p_2 = 1$). Let $N_t = M_t - 1$ be the number of cells produced in the solution by time t.

Calculate the distribution of N_t. Is (N_t) an inhomogeneous Poisson process? If so, what is its rate $\lambda(t)$? Justify your answer.

2/II/26I Applied Probability

What does it mean to say that (X_t) is a renewal process?

Let (X_t) be a renewal process with holding times $S_1, S_2, \ldots$ and let $s > 0$. For $n \geqslant 1$, set $T_n = S_{X_s + n}$. Show that

$$\mathbb{P}(T_n > t) \geqslant \mathbb{P}(S_n > t), \quad t \geqslant 0,$$

for all n, with equality if $n \geqslant 2$.

Consider now the case where $S_1, S_2, \ldots$ are exponential random variables. Show that

$$\mathbb{P}(T_1 > t) > \mathbb{P}(S_1 > t), \quad t > 0,$$

and that, as $s \to \infty$,

$$\mathbb{P}(T_1 > t) \to \mathbb{P}(S_1 + S_2 > t), \quad t \geqslant 0\,.$$

3/II/25I **Applied Probability**

Consider an M/G/r/0 loss system with arrival rate λ and service-time distribution F. Thus, arrivals form a Poisson process of rate λ, service times are independent with common distribution F, there are r servers and there is no space for waiting. Use Little's Lemma to obtain a relation between the long-run average occupancy L and the stationary probability π that the system is full.

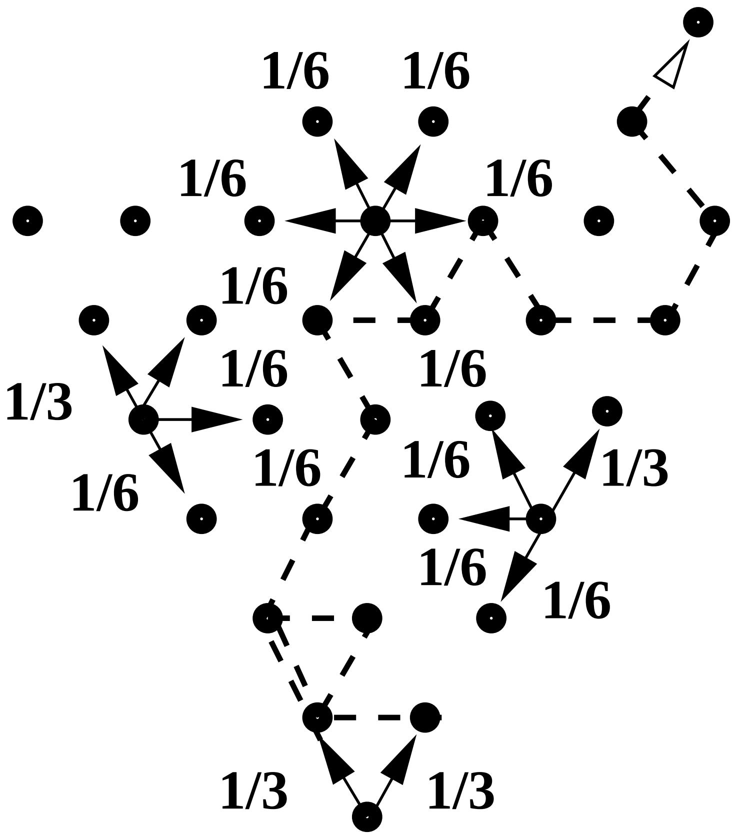
Cafe–Bar Duo has 23 serving tables. Each table can be occupied either by one person or two. Customers arrive either singly or in a pair; if a table is empty they are seated and served immediately, otherwise, they leave. The times between arrivals are independent exponential random variables of mean 20/3. Each arrival is twice as likely to be a single person as a pair. A single customer stays for an exponential time of mean 20, whereas a pair stays for an exponential time of mean 30; all these times are independent of each other and of the process of arrivals. The value of orders taken at each table is a constant multiple 2/5 of the time that it is occupied.

Express the long-run rate of revenue of the cafe as a function of the probability π that an arriving customer or pair of customers finds the cafe full.

By imagining a cafe with infinitely many tables, show that $\pi \leqslant \mathbb{P}(N \geqslant 23)$ where N is a Poisson random variable of parameter 7/2. Deduce that π is very small. [*Credit will be given for any useful numerical estimate, an upper bound of* 10^{-3} *being sufficient for full credit.*]

4/II/26I **Applied Probability**

A particle performs a continuous-time nearest neighbour random walk on a regular triangular lattice inside an angle $\pi/3$, starting from the corner. See the diagram below. The jump rates are 1/3 from the corner and 1/6 in each of the six directions if the particle is inside the angle. However, if the particle is on the edge of the angle, the rate is 1/3 along the edge away from the corner and 1/6 to each of three other neighbouring sites in the angle. See the diagram below, where a typical trajectory is also shown.



The particle position at time $t \geqslant 0$ is determined by its vertical level V_t and its horizontal position G_t. For $k \geqslant 0$, if $V_t = k$ then $G_t = 0, \ldots, k$. Here $1, \ldots, k-1$ are positions inside, and 0 and k positions on the edge of the angle, at vertical level k.

Let J_1^V, J_2^V, ... be the times of subsequent jumps of process (V_t) and consider the embedded discrete-time Markov chains

$$Y_n^{\mathrm{in}} = \left(\widehat{G}_n^{\mathrm{in}}, \widehat{V}_n\right) \text{ and } Y_n^{\mathrm{out}} = \left(\widehat{G}_n^{\mathrm{out}}, \widehat{V}_n\right)$$

where $\widehat{V}_n$ is the vertical level immediately after time J_n^V, $\widehat{G}_n^{\mathrm{in}}$ is the horizontal position immediately after time J_n^V, and $\widehat{G}_n^{\mathrm{out}}$ is the horizontal position immediately before time J_{n+1}^V.

(a) Assume that $(\widehat{V}_n)$ is a Markov chain with transition probabilities

$$\mathbb{P}\big(\widehat{V}_n = k+1 \big| \widehat{V}_{n-1} = k\big) = \frac{k+2}{2(k+1)}, \;\; \mathbb{P}\big(\widehat{V}_n = k-1 \big| \widehat{V}_{n-1} = k\big) = \frac{k}{2(k+1)},$$

and that (V_t) is a continuous-time Markov chain with rates

$$q_{kk-1} = \frac{k}{3(k+1)}, \;\; q_{kk} = -\frac{2}{3}, \;\; q_{kk+1} = \frac{k+2}{3(k+1)}.$$

[*You will be asked to justify these assumptions in part (b) of the question.*] Determine whether the chains $(\widehat{V}_n)$ and (V_t) are transient, positive recurrent or null recurrent.

(b) Now assume that, conditional on $\widehat{V}_n = k$ and previously passed vertical levels, the horizontal positions $\widehat{G}_n^{\rm in}$ and $\widehat{G}_n^{\rm out}$ are uniformly distributed on $\{0, \ldots, k\}$. In other words, for all attainable values k, k_{n-1}, …, k_1 and for all $i = 0, \ldots, k$,

$$\begin{aligned} &\mathbb{P}\big(\widehat{G}_n^{\rm in} = i \big| \widehat{V}_n = k, \widehat{V}_{n-1} = k_{n-1}, \ldots, \widehat{V}_1 = k_1, \widehat{V}_0 = 0\big) \\ &\quad = \mathbb{P}\big(\widehat{G}_n^{\rm out} = i \big| \widehat{V}_n = k, \widehat{V}_{n-1} = k_{n-1}, \ldots, \widehat{V}_1 = k_1, \widehat{V}_0 = 0\big) = \frac{1}{k+1}\,. \end{aligned} \qquad (*)$$

Deduce that $(\widehat{V}_n)$ and (V_t) are indeed Markov chains with transition probabilities and rates as in (a).

(c) Finally, prove property $(*)$.

1/II/27I Principles of Statistics

State *Wilks' Theorem* on the asymptotic distribution of likelihood-ratio test statistics.

Suppose that $X_1, \ldots, X_n$ are independent with common $N(\mu, \sigma^2)$ distribution, where the parameters μ and σ are both unknown. Find the likelihood-ratio test statistic for testing $H_0 : \mu = 0$ against $H_1 : \mu$ unrestricted, and state its (approximate) distribution.

What is the form of the t-test of H_0 against H_1? Explain why for large n the likelihood-ratio test and the t-test are nearly the same.

2/II/27I Principles of Statistics

(i) Suppose that X is a multivariate normal vector with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\sigma^2 I$, where μ and σ^2 are both unknown, and I denotes the $d \times d$ identity matrix. Suppose that $\Theta_0 \subset \Theta_1$ are linear subspaces of $\mathbb{R}^d$ of dimensions d_0 and d_1, where $d_0 < d_1 < d$. Let P_i denote orthogonal projection onto Θ_i $(i = 0, 1)$. Carefully derive the joint distribution of $(|X - P_1 X|^2, |P_1 X - P_0 X|^2)$ under the hypothesis $H_0 : \mu \in \Theta_0$. How could you use this to make a test of H_0 against $H_1 : \mu \in \Theta_1$?

(ii) Suppose that I students take J exams, and that the mark X_{ij} of student i in exam j is modelled as

$$X_{ij} = m + \alpha_i + \beta_j + \varepsilon_{ij}$$

where $\sum_i \alpha_i = 0 = \sum_j \beta_j$, the ε_{ij} are independent $N(0, \sigma^2)$, and the parameters m, α, β and σ are unknown. Construct a test of $H_0 : \beta_j = 0$ for all j against $H_1 : \sum_j \beta_j = 0$.

3/II/26I Principles of Statistics

In the context of decision theory, explain the meaning of the following italicized terms: *loss function, decision rule*, the *risk* of a decision rule, a *Bayes rule* with respect to prior π, and an *admissible* rule. Explain how a Bayes rule with respect to a prior π can be constructed.

Suppose that $X_1, \ldots, X_n$ are independent with common $N(0, v)$ distribution, where $v > 0$ is supposed to have a prior density f_0. In a decision-theoretic approach to estimating v, we take a quadratic loss: $L(v, a) = (v - a)^2$. Write $X = (X_1, \ldots, X_n)$ and $|X| = (X_1^2 + \ldots + X_n^2)^{1/2}$.

By considering decision rules (estimators) of the form $\hat{v}(X) = \alpha|X|^2$, prove that if $\alpha \neq 1/(n+2)$ then the estimator $\hat{v}(X) = \alpha|X|^2$ is not Bayes, for any choice of prior f_0.

By considering decision rules of the form $\hat{v}(X) = \alpha|X|^2 + \beta$, prove that if $\alpha \neq 1/n$ then the estimator $\hat{v}(X) = \alpha|X|^2$ is not Bayes, for any choice of prior f_0.

[*You may use without proof the fact that, if Z has a $N(0,1)$ distribution, then $EZ^4 = 3$.*]

4/II/27I **Principles of Statistics**

A group of n hospitals is to be 'appraised'; the 'performance' θ_i of hospital i has a $N(0, 1/\tau)$ prior distribution, different hospitals being independent. The 'performance' cannot be measured directly, so an expensive firm of management consultants has been hired to arrive at each hospital's Standardised Index of Quality [SIQ], this being a number X_i for hospital i related to θ_i by the commercially-sensitive formula

$$X_i = \theta_i + \varepsilon_i,$$

where the ε_i are independent with common $N(0, 1/\tau_\varepsilon)$ distribution.

(i) Assume that τ and τ_ε are known. What is the posterior distribution of θ given X? Suppose that hospital j was the hospital with the lowest SIQ, with a value $X_j = x$; conditional on X, what is the distribution of θ_j?

(ii) Now, instead of assuming τ and τ_ε known, suppose that τ has a Gamma prior with parameters (α, β), density

$$f(t) = (\beta t)^{\alpha-1}\beta e^{-\beta t}/\Gamma(\alpha)$$

for known α and β, and that $\tau_\varepsilon = \kappa\tau$, where κ is a known constant. Find the posterior distribution of (θ, τ) given X. Comment briefly on the form of the distribution.

1/II/28J **Stochastic Financial Models**

Let $X \equiv (X_0, X_1, \ldots, X_J)^T$ be a zero-mean Gaussian vector, with covariance matrix $V = (v_{jk})$. In a simple single-period economy with J agents, agent i will receive X_i at time 1 ($i = 1, \ldots, J$). If Y is a contingent claim to be paid at time 1, define agent i's *reservation bid price* for Y, assuming his preferences are given by $E[U_i(X_i + Z)]$ for any contingent claim Z.

Assuming that $U_i(x) \equiv -\exp(-\gamma_i x)$ for each i, where $\gamma_i > 0$, show that agent i's reservation bid price for λ units of X_0 is

$$p_i(\lambda) = -\frac{1}{2}\gamma_i(\lambda^2 v_{00} + 2\lambda v_{0i}).$$

As $\lambda \to 0$, find the limit of agent i's per-unit reservation bid price for X_0, and comment on the expression you obtain.

The agents bargain, and reach an equilibrium. Assuming that the contingent claim X_0 is in zero net supply, show that the equilibrium price of X_0 will be

$$p = -\Gamma v_{0\bullet},$$

where $\Gamma^{-1} = \sum_{i=1}^{J} \gamma_i^{-1}$ and $v_{0\bullet} = \sum_{i=1}^{J} v_{0i}$. Show that at that price agent i will choose to buy

$$\theta_i = (\Gamma v_{0\bullet} - \gamma_i v_{0i})/(\gamma_i v_{00})$$

units of X_0.

By computing the improvement in agent i's expected utility, show that the value to agent i of access to this market is equal to a fixed payment of

$$\frac{(\gamma_i v_{0i} - \Gamma v_{0\bullet})^2}{2\gamma_i v_{00}}.$$

2/II/28J Stochastic Financial Models

(i) At the beginning of year n, an investor makes decisions about his investment and consumption for the coming year. He first takes out an amount c_n from his current wealth w_n, and sets this aside for consumption. He splits his remaining wealth between a bank account (unit wealth invested at the start of the year will have grown to a sure amount $r > 1$ by the end of the year), and the stock market. Unit wealth invested in the stock market will have become the random amount $X_{n+1} > 0$ by the end of the year.

The investor's objective is to invest and consume so as to maximise the expected value of $\sum_{n=1}^{N} U(c_n)$, where U is strictly increasing and strictly convex. Consider the dynamic programming equation (Bellman equation) for his problem,

$$V_n(w) = \sup_{c,\theta} \left\{ U(c) + E_n\left[V_{n+1}(\theta(w-c)X_{n+1} + (1-\theta)(w-c)r) \right] \right\} \quad (0 \leqslant n < N),$$
$$V_N(w) = U(w).$$

Explain all undefined notation, and explain briefly why the equation holds.

(ii) Supposing that the X_i are independent and identically distributed, and that $U(x) = x^{1-R}/(1-R)$, where $R > 0$ is different from 1, find as explicitly as you can the form of the agent's optimal policy.

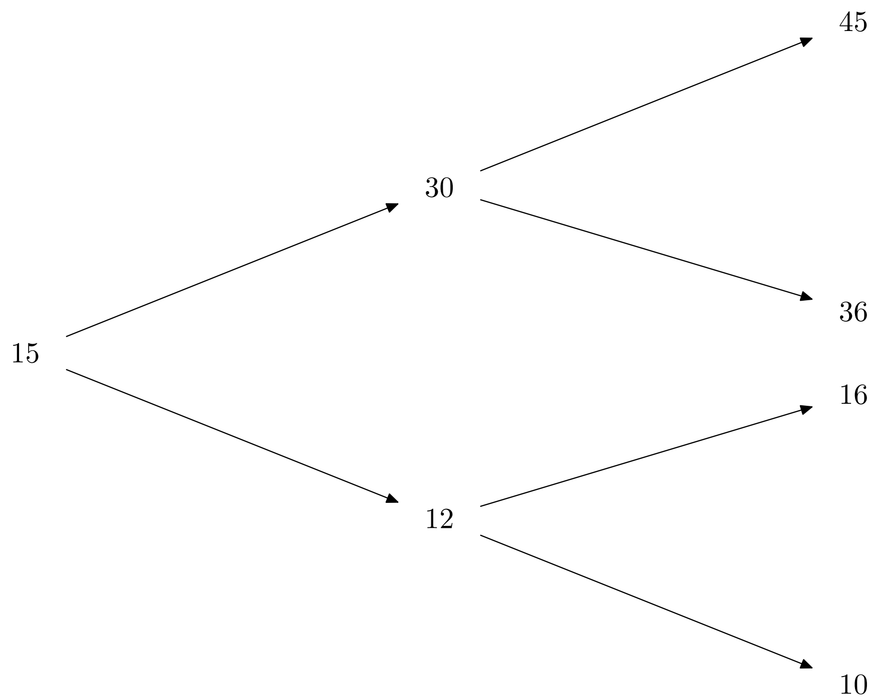
(iii) Return to the general problem of (i). Assuming that the sample space Ω is finite, and that all suprema are attained, show that

$$\begin{aligned} E_n[\, V'_{n+1}(w^*_{n+1})(X_{n+1} - r)\,] &= 0, \\ rE_n[\, V'_{n+1}(w^*_{n+1})\,] &= U'(c^*_n), \\ rE_n[\, V'_{n+1}(w^*_{n+1})\,] &= V'_n(w^*_n), \end{aligned}$$

where $(c^*_n, w^*_n)_{0 \leqslant n \leqslant N}$ denotes the optimal consumption and wealth process for the problem. Explain the significance of each of these equalities.

3/II/27J **Stochastic Financial Models**

Suppose that over two periods a stock price moves on a binomial tree:



(a) Find an arbitrage opportunity when the riskless rate equals 1/10. Give precise details of when and how much you buy, borrow and sell.

(b) From here on, assume instead that the riskless rate equals 1/4. Determine the equivalent martingale measure. [*No proof is required.*]

(c) Determine the time-zero price of an American put with strike 15 and expiry 2. Assume you sell it at this price. Which hedge do you put on at time zero? Consider the scenario of two bad periods. How does your hedge work?

(d) The buyer of the American put turns out to be an unsophisticated investor who fails to use his early exercise right when he should. Assume the first period was bad. How much profit can you make out of this? You should detail your exact strategy.

4/II/28J **Stochastic Financial Models**

(a) In the context of the Black–Scholes formula, let S_0 be spot price, K be strike price, T be time to maturity, and assume constant interest rate r, volatility σ and absence of dividends. Write down explicitly the prices of a European call and put,

$$EC\left(S_0, K, \sigma, r, T\right) \text{ and } EP\left(S_0, K, \sigma, r, T\right).$$

Use Φ for the normal distribution function. [*No proof is required.*]

(b) From here on assume $r = 0$. Keeping T, σ fixed, we shorten the notation to $EC\left(S_0, K\right)$ and similarly for EP. Show that *put-call symmetry* holds:

$$EC\left(S_0, K\right) = EP\left(K, S_0\right).$$

Check homogeneity: for every real $\alpha > 0$

$$EC\left(\alpha S_0, \alpha K\right) = \alpha EC\left(S_0, K\right).$$

(c) Show that the price of a down-and-out European call with strike $K < S_0$ and barrier $B \leqslant K$ is given by

$$EC\left(S_0, K\right) - \frac{S_0}{B} EC\left(\frac{B^2}{S_0}, K\right).$$

(d)

(i) Specialize the last expression to $B = K$ and simplify.

(ii) Answer a popular interview question in investment banks: What is the fair value of a down-and-out call given that $S_0 = 100$, $B = K = 80$, $\sigma = 20\%$, $r = 0$, $T = 1$? Identify the corresponding hedge. [*It may be helpful to compute Delta first.*]

(iii) Does this hedge work beyond the Black–Scholes model? When does it fail?

2/II/29I Optimization and Control

Explain what is meant by a time-homogeneous discrete time Markov decision problem.

What is the positive programming case?

A discrete time Markov decision problem has state space $\{0, 1, \ldots, N\}$. In state i, $i \neq 0, N$, two actions are possible. We may either stop and obtain a terminal reward $r(i) \geqslant 0$, or may continue, in which case the subsequent state is equally likely to be $i-1$ or $i+1$. In states 0 and N stopping is automatic (with terminal rewards $r(0)$ and $r(N)$ respectively). Starting in state i, denote by $V_n(i)$ and $V(i)$ the maximal expected terminal reward that can be obtained over the first n steps and over the infinite horizon, respectively. Prove that $\lim_{n\to\infty} V_n = V$.

Prove that V is the smallest concave function such that $V(i) \geqslant r(i)$ for all i.

Describe an optimal policy.

Suppose $r(0), \ldots, r(N)$ are distinct numbers. Show that the optimal policy is unique, or give a counter-example.

3/II/28I Optimization and Control

Consider the problem

$$\text{minimize } E\left[x(T)^2 + \int_0^T u(t)^2\, dt\right]$$

where for $0 \leqslant t \leqslant T$,

$$\dot{x}(t) = y(t) \quad \text{and} \quad \dot{y}(t) = u(t) + \epsilon(t)\,,$$

$u(t)$ is the control variable, and $\epsilon(t)$ is Gaussian white noise. Show that the problem can be rewritten as one of controlling the scalar variable $z(t)$, where

$$z(t) = x(t) + (T-t)y(t)\,.$$

By guessing the form of the optimal value function and ensuring it satisfies an appropriate optimality equation, show that the optimal control is

$$u(t) = -\frac{(T-t)z(t)}{1 + \frac{1}{3}(T-t)^3}\,.$$

Is this certainty equivalence control?

4/II/29I **Optimization and Control**

A continuous-time control problem is defined in terms of state variable $x(t) \in \mathbb{R}^n$ and control $u(t) \in \mathbb{R}^m$, $0 \leqslant t \leqslant T$. We desire to minimize $\int_0^T c(x,t)\,dt + K(x(T))$, where T is fixed and $x(T)$ is unconstrained. Given $x(0)$ and $\dot{x} = a(x,u)$, describe further boundary conditions that can be used in conjunction with Pontryagin's maximum principle to find x, u and the adjoint variables $\lambda_1, \ldots, \lambda_n$.

Company 1 wishes to steal customers from Company 2 and maximize the profit it obtains over an interval $[0,T]$. Denoting by $x_i(t)$ the number of customers of Company i, and by $u(t)$ the advertising effort of Company 1, this leads to a problem

$$\text{minimize} \int_0^T \big[x_2(t) + 3u(t)\big]\,dt\,,$$

where $\dot{x}_1 = ux_2$, $\dot{x}_2 = -ux_2$, and $u(t)$ is constrained to the interval $[0,1]$. Assuming $x_2(0) > 3/T$, use Pontryagin's maximum principle to show that the optimal advertising policy is bang-bang, and that there is just one change in advertising effort, at a time t^*, where

$$3\,e^{t^*} = x_2(0)(T - t^*)\,.$$

1/II/29C Partial Differential Equations

Consider the equation

$$x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2} + a \frac{\partial u}{\partial x_3} = u, \qquad (*)$$

where $a \in \mathbb{R}$, to be solved for $u = u(x_1, x_2, x_3)$. State clearly what it means for a hypersurface

$$S_\phi = \{(x_1, x_2, x_3) : \ \phi(x_1, x_2, x_3) = 0\},$$

defined by a C^1 function ϕ, to be *non-characteristic for* $(*)$. Does the non-characteristic condition hold when $\phi(x_1, x_2, x_3) = x_3$?

Solve $(*)$ for $a > 0$ with initial condition $u(x_1, x_2, 0) = f(x_1, x_2)$ where $f \in C^1(\mathbb{R}^2)$. For the case $f(x_1, x_2) = x_1^2 + x_2^2$ discuss the limiting behaviour as $a \to 0_+$.

2/II/30C Partial Differential Equations

Define a *fundamental solution* of a linear partial differential operator P. Prove that the function

$$G(x) = \tfrac{1}{2} e^{-|x|}$$

defines a distribution which is a fundamental solution of the operator P given by

$$P\,u = -\frac{d^2 u}{dx^2} + u\,.$$

Hence find a solution u_0 to the equation

$$-\frac{d^2 u_0}{dx^2} + u_0 = V(x)\,,$$

where $V(x) = 0$ for $|x| > 1$ and $V(x) = 1$ for $|x| \leqslant 1$.

Consider the functional

$$I[u] = \int_{\mathbb{R}} \left\{ \frac{1}{2}\Big[\Big(\frac{du}{dx}\Big)^2 + u^2\Big] - V u \right\} dx\,.$$

Show that $I[u_0 + \phi] > I[u_0]$ for all Schwartz functions ϕ that are not identically zero.

3/II/29C Partial Differential Equations

Write down a formula for the solution $u = u(t, x)$ of the n-dimensional heat equation

$$w_t(t, x) - \Delta w = 0, \qquad w(0, x) = g(x),$$

for $g : \mathbb{R}^n \to \mathbb{C}$ a given Schwartz function; here $w_t = \partial_t w$ and Δ is taken in the variables $x \in \mathbb{R}^n$. Show that

$$w(t, x) \leqslant \frac{\int |g(x)|\, dx}{(4\pi t)^{n/2}}\,.$$

Consider the equation

$$u_t - \Delta u = e^{it} f(x)\,, \qquad (*)$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a given Schwartz function. Show that $(*)$ has a solution of the form

$$u(t, x) = e^{it} v(x)\,,$$

where v is a Schwartz function.

Prove that the solution $u(t, x)$ of the initial value problem for $(*)$ with initial data $u(0, x) = g(x)$ satisfies

$$\lim_{t \to +\infty} \left| u(t, x) - e^{it} v(x) \right| = 0\,.$$

4/II/30C Partial Differential Equations

Write down the solution of the three-dimensional wave equation

$$u_{tt} - \Delta u = 0\,, \qquad u(0, x) = 0\,, \quad u_t(0, x) = g(x)\,,$$

for a Schwartz function g. Here Δ is taken in the variables $x \in \mathbb{R}^3$ and $u_t = \partial u / \partial t$ etc. State the "strong" form of Huygens principle for this solution. Using the method of descent, obtain the solution of the corresponding problem in two dimensions. State the "weak" form of Huygens principle for this solution.

Let $u \in C^2([0, T] \times \mathbb{R}^3)$ be a solution of

$$u_{tt} - \Delta u + |x|^2 u = 0\,, \qquad u(0, x) = 0\,, \quad u_t(0, x) = 0\,. \qquad (*)$$

Show that

$$\partial_t e + \boldsymbol{\nabla} \cdot \mathbf{p} = 0\,, \qquad (**)$$

where

$$e = \tfrac{1}{2}\big(u_t{}^2 + |\boldsymbol{\nabla} u|^2 + |x|^2 u^2\big), \qquad \text{and} \qquad \mathbf{p} = -u_t \boldsymbol{\nabla} u\,.$$

Hence deduce, by integration of $(**)$ over the region

$$K = \big\{(t, x) : 0 \leqslant t \leqslant t_0 - a \leqslant t_0,\ \ |x - x_0| \leqslant t_0 - t\big\}$$

or otherwise, that $(*)$ satisfies the weak Huygens principle.

1/II/30A **Asymptotic Methods**

Explain what is meant by an asymptotic power series about $x = a$ for a real function $f(x)$ of a real variable. Show that a convergent power series is also asymptotic.

Show further that an asymptotic power series is unique (assuming that it exists).

Let the function $f(t)$ be defined for $t \geqslant 0$ by

$$f(t) = \frac{1}{\pi^{1/2}} \int_0^\infty \frac{e^{-x}}{x^{1/2}(1 + 2xt)} dx \, .$$

By suitably expanding the denominator of the integrand, or otherwise, show that, as $t \to 0_+$,

$$f(t) \sim \sum_{k=0}^{\infty} (-1)^k 1.3 \ldots (2k-1) t^k$$

and that the error, when the series is stopped after n terms, does not exceed the absolute value of the $(n+1)$th term of the series.

3/II/30A **Asymptotic Methods**

Explain, without proof, how to obtain an asymptotic expansion, as $x \to \infty$, of

$$I(x) = \int_0^\infty e^{-xt} f(t) dt \, ,$$

if it is known that $f(t)$ possesses an asymptotic power series as $t \to 0$.

Indicate the modification required to obtain an asymptotic expansion, under suitable conditions, of

$$\int_{-\infty}^{\infty} e^{-xt^2} f(t) \; dt \, .$$

Find an asymptotic expansion as $z \to \infty$ of the function defined by

$$I(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(z - t)} dt \qquad (\mathrm{Im}(z) < 0)$$

and its analytic continuation to $\mathrm{Im}(z) \geqslant 0$. Where are the Stokes lines, that is, the critical lines separating the Stokes regions?

4/II/31A **Asymptotic Methods**

Consider the differential equation

$$\frac{d^2w}{dx^2} = q(x)w\,,$$

where $q(x) \geqslant 0$ in an interval (a, ∞). Given a solution $w(x)$ and a further smooth function $\xi(x)$, define

$$W(x) = [\xi'(x)]^{1/2} w(x)\,.$$

Show that, when ξ is regarded as the independent variable, the function $W(\xi)$ obeys the differential equation

$$\frac{d^2W}{d\xi^2} = \left\{ \dot{x}^2 q(x) + \dot{x}^{1/2} \frac{d^2}{d\xi^2}[\dot{x}^{-1/2}] \right\} W, \tag{$*$}$$

where $\dot{x}$ denotes $dx/d\xi$.

Taking the choice

$$\xi(x) = \int q^{1/2}(x)dx\,,$$

show that equation $(*)$ becomes

$$\frac{d^2W}{d\xi^2} = (1+\phi)W\,,$$

where

$$\phi = -\frac{1}{q^{3/4}} \frac{d^2}{dx^2}\Big(\frac{1}{q^{1/4}}\Big)\,.$$

In the case that ϕ is negligible, deduce the Liouville–Green approximate solutions

$$w_{\pm} = q^{-1/4} \exp\Big(\pm \int q^{1/2} dx\Big)\,.$$

Consider the Whittaker equation

$$\frac{d^2w}{dx^2} = \left[\frac{1}{4} + \frac{s(s-1)}{x^2}\right] w\,,$$

where s is a real constant. Show that the Liouville–Green approximation suggests the existence of solutions $w_{A,B}(x)$ with asymptotic behaviour of the form

$$w_A \sim \exp(x/2)\left(1 + \sum_{n=1}^{\infty} a_n x^{-n}\right), \qquad w_B \sim \exp(-x/2)\left(1 + \sum_{n=1}^{\infty} b_n x^{-n}\right)$$

as $x \to \infty$.

Given that these asymptotic series may be differentiated term-by-term, show that

$$a_n = \frac{(-1)^n}{n!}(s-n)(s-n+1)\ldots(s+n-1)\,.$$

1/II/31D Integrable Systems

Let $\phi(t)$ satisfy the linear singular integral equation

$$(t^2+t-1)\phi(t) - \frac{t^2-t-1}{\pi i} \oint_L \frac{\phi(\tau)d\tau}{\tau - t} - \frac{1}{\pi i}\int_L \left(\tau + \frac{1}{\tau}\right)\phi(\tau)d\tau = t-1, \quad t \in L,$$

where $\oint$ denotes the principal value integral and L denotes a counterclockwise smooth closed contour, enclosing the origin but not the points ± 1.

(a) Formulate the associated Riemann–Hilbert problem.

(b) For this Riemann–Hilbert problem, find the index, the homogeneous canonical solution and the solvability condition.

(c) Find $\phi(t)$.

2/II/31C Integrable Systems

Suppose $q(x,t)$ satisfies the mKdV equation

$$q_t + q_{xxx} + 6q^2 q_x = 0\,,$$

where $q_t = \partial q/\partial t$ etc.

(a) Find the 1-soliton solution.

[*You may use, without proof, the indefinite integral* $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{arcsech} x\,.$]

(b) Express the self-similar solution of the mKdV equation in terms of a solution, denoted by $v(z)$, of the Painlevé II equation.

(c) Using the Ansatz

$$\frac{dv}{dz} + iv^2 - \frac{i}{6}z = 0\,,$$

find a particular solution of the mKdV equation in terms of a solution of the Airy equation

$$\frac{d^2\Psi}{dz^2} + \frac{z}{6}\Psi = 0\,.$$

3/II/31A Integrable Systems

Let $Q(x,t)$ be an off-diagonal 2×2 matrix. The matrix NLS equation

$$iQ_t - Q_{xx}\sigma_3 + 2Q^3\sigma_3 = 0, \quad \sigma_3 = diag(1,-1),$$

admits the Lax pair

$$\begin{aligned} &\mu_x + ik[\sigma_3, \mu] = Q\mu, \\ &\mu_t + 2ik^2[\sigma_3, \mu] = (2kQ - iQ^2\sigma_3 - iQ_x\sigma_3)\mu, \end{aligned}$$

where $k \in \mathbb{C}$, $\mu(x,t,k)$ is a 2×2 matrix and $[\sigma_3, \mu]$ denotes the matrix commutator.

Let $S(k)$ be a 2×2 matrix-valued function decaying as $|k| \to \infty$. Let $\mu(x,t,k)$ satisfy the 2×2-matrix Riemann–Hilbert problem

$$\mu^+(x,t,k) = \mu^-(x,t,k)e^{-i(kx+2k^2t)\sigma_3}S(k)e^{i(kx+2k^2t)\sigma_3}, \quad k \in \mathbb{R},$$

$$\mu = diag(1,1) + \mathrm{O}\left(\frac{1}{k}\right), \quad k \to \infty.$$

(a) Find expressions for $Q(x,t)$, $A(x,t)$ and $B(x,t)$, in terms of the coefficients in the large k expansion of μ, so that μ solves

$$\mu_x + ik[\sigma_3, \mu] - Q\mu = 0,$$

and

$$\mu_t + 2ik^2[\sigma_3, \mu] - (kA + B)\mu = 0.$$

(b) Use the result of (a) to establish that

$$A = 2Q, \quad B = -i(Q^2 + Q_x)\sigma_3.$$

(c) Show that the above results provide a linearization of the matrix NLS equation. What is the disadvantage of this approach in comparison with the inverse scattering method?

1/II/32D Principles of Quantum Mechanics

A one-dimensional harmonic oscillator has Hamiltonian

$$H \;=\; \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 \;=\; \hbar\omega\Big(a^\dagger a \,+\, \frac{1}{2}\Big)\,,$$

where

$$a = \Big(\frac{m\omega}{2\hbar}\Big)^{1/2}\Big(\hat{x} + \frac{i}{m\omega}\hat{p}\Big)\ , \quad a^\dagger = \Big(\frac{m\omega}{2\hbar}\Big)^{1/2}\Big(\hat{x} - \frac{i}{m\omega}\hat{p}\Big) \;\text{ obey }\; [a, a^\dagger] = 1.$$

Assuming the existence of a normalised state $|0\rangle$ with $a|0\rangle = 0$, verify that

$$|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger\, n}|0\rangle\ , \qquad n = 0, 1, 2, \ldots$$

are eigenstates of H with energies E_n, to be determined, and that these states all have unit norm.

The Hamiltonian is now modified by a term

$$\lambda V = \lambda\hbar\omega(\, a^r + a^{\dagger\, r}\,)$$

where r is a positive integer. Use perturbation theory to find the change in the lowest energy level to order λ^2 for any r. [*You may quote any standard formula you need.*]

Compute by perturbation theory, again to order λ^2, the change in the first excited energy level when $r = 1$. Show that in this special case, $r = 1$, the *exact* change in *all* energy levels as a result of the perturbation is $-\lambda^2\hbar\omega$.

2/II/32D Principles of Quantum Mechanics

The components of $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are 2×2 hermitian matrices obeying

$$[\,\sigma_i,\, \sigma_j\,] \;=\; 2i\varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1 \qquad (*)$$

for any unit vector $\mathbf{n}$. Show that these properties imply

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\,(\mathbf{b}\cdot\boldsymbol{\sigma}) \;=\; \mathbf{a}\cdot\mathbf{b} \,+\, i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}$$

for any constant vectors $\mathbf{a}$ and $\mathbf{b}$. Assuming that θ is real, explain why the matrix $U = \exp(-i\mathbf{n}\cdot\boldsymbol{\sigma}\,\theta/2)$ is unitary, and show that

$$U \;=\; \cos(\theta/2) \,-\, i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta/2)\;.$$

Hence deduce that

$$U\mathbf{m}\cdot\boldsymbol{\sigma}U^{-1} \;=\; \mathbf{m}\cdot\boldsymbol{\sigma}\cos\theta \,+\, (\mathbf{n}\times\mathbf{m})\cdot\boldsymbol{\sigma}\sin\theta$$

where $\mathbf{m}$ is any unit vector orthogonal to $\mathbf{n}$.

Write down an equation relating the matrices $\boldsymbol{\sigma}$ and the angular momentum operator $\mathbf{S}$ for a particle of spin one half, and explain *briefly* the significance of the conditions $(*)$. Show that if $|\chi\rangle$ is a state with spin 'up' measured along the direction $(0,0,1)$ then, for a certain choice of $\mathbf{n}$, $U|\chi\rangle$ is a state with spin 'up' measured along the direction $(\sin\theta, 0, \cos\theta)$.

3/II/32D Principles of Quantum Mechanics

The angular momentum operators $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ refer to independent systems, each with total angular momentum one. The combination of these systems has a basis of states which are of product form $|m_1; m_2\rangle = |1\,m_1\rangle|1\,m_2\rangle$ where m_1 and m_2 are the eigenvalues of $J_3^{(1)}$ and $J_3^{(2)}$ respectively. Let $|J\,M\rangle$ denote the alternative basis states which are simultaneous eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$ is the combined angular momentum. What are the possible values of J and M? Find expressions for all states with $J = 1$ in terms of product states. How do these states behave when the constituent systems are interchanged?

Two spin-one particles A and B have no mutual interaction but they each move in a potential $V(\mathbf{r})$ which is independent of spin. The single-particle energy levels E_i and the corresponding wavefunctions $\psi_i(\mathbf{r})$ $(i = 1, 2, \ldots)$ are the same for either A or B. Given that $E_1 < E_2 < \ldots$, explain how to construct the two-particle states of lowest energy and combined total spin $J = 1$ for the cases that (i) A and B are identical, and (ii) A and B are not identical.

[*You may assume* $\hbar = 1$ *and use the result* $J_\pm|j\,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j\;m\pm 1\rangle$.]

4/II/32D **Principles of Quantum Mechanics**

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + \lambda V(t)\,,$$

where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|a\rangle$ and $|b\rangle$ be eigenstates of H_0 with distinct eigenvalues E_a and E_b respectively. Show that if the system is initially in state $|a\rangle$ then the probability of measuring it to be in state $|b\rangle$ after a time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_0^t dt' \langle b|V(t')|a\rangle e^{i(E_b-E_a)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\ .$$

Deduce that if $V(t) = e^{-\mu t/\hbar}W$, where W is a time-independent operator and μ is a positive constant, then the probability for such a transition to have occurred after a very long time is approximately

$$\frac{\lambda^2}{\mu^2 + (E_b - E_a)^2}|\langle b|W|a\rangle|^2\ .$$

1/II/33B **Applications of Quantum Mechanics**

A beam of particles is incident on a central potential $V(r)$ ($r = |\mathbf{x}|$) that vanishes for $r > R$. Define the differential cross-section $d\sigma/d\Omega$.

Given that each incoming particle has momentum $\hbar\mathbf{k}$, explain the relevance of solutions to the time-independent Schrödinger equation with the asymptotic form

$$\psi(\mathbf{x}) \sim e^{i\mathbf{k}\cdot\mathbf{x}} + f(\hat{\mathbf{x}})\,\frac{e^{ikr}}{r} \qquad (*)$$

as $r \to \infty$, where $k = |\mathbf{k}|$ and $\hat{\mathbf{x}} = \mathbf{x}/r$. Write down a formula that determines $d\sigma/d\Omega$ in this case.

Write down the time-independent Schrödinger equation for a particle of mass m and energy $E = \dfrac{\hbar^2 k^2}{2m}$ in a central potential $V(r)$, and show that it allows a solution of the form

$$\psi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{m}{2\pi\hbar^2}\int d^3x'\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,V(r')\psi(\mathbf{x}')\;.$$

Show that this is consistent with $(*)$ and deduce an expression for $f(\hat{\mathbf{x}})$. Obtain the Born approximation for $f(\hat{\mathbf{x}})$, and show that $f(\hat{\mathbf{x}}) = F(k\hat{\mathbf{x}} - \mathbf{k})$, where

$$F(\mathbf{q}) = -\frac{m}{2\pi\hbar^2}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\,V(r)\,.$$

Under what conditions is the Born approximation valid?

Obtain a formula for $f(\hat{\mathbf{x}})$ in terms of the scattering angle θ in the case that

$$V(r) = K\,\frac{e^{-\mu r}}{r}\,,$$

for constants K and μ. Hence show that $f(\hat{\mathbf{x}})$ is independent of $\hbar$ in the limit $\mu \to 0$, when expressed in terms of θ and the energy E.

[*You may assume that* $(\nabla^2 + k^2)\left(\dfrac{e^{ikr}}{r}\right) = -4\pi\delta^3(\mathbf{x})$.]

2/II/33B **Applications of Quantum Mechanics**

Describe briefly the variational approach to the determination of an approximate ground state energy E_0 of a Hamiltonian H.

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two states, and consider the trial state

$$|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle$$

for real constants a_1 and a_2. Given that

$$\begin{aligned} \langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1\,, &\qquad \langle\psi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle = s\,, \\ \langle\psi_1|H|\psi_1\rangle = \langle\psi_2|H|\psi_2\rangle = \mathcal{E}\,, &\quad \langle\psi_2|H|\psi_1\rangle = \langle\psi_1|H|\psi_2\rangle = \epsilon\,, \end{aligned} \tag{$*$}$$

and that $\epsilon < s\mathcal{E}$, obtain an upper bound on E_0 in terms of $\mathcal{E}$, ϵ and s.

The normalized ground-state wavefunction of the Hamiltonian

$$H_1 = \frac{p^2}{2m} - K\delta(x)\,, \qquad K > 0,$$

is

$$\psi_1(x) = \sqrt{\lambda}\, e^{-\lambda|x|}\,, \qquad \lambda = \frac{mK}{\hbar^2}\,.$$

Verify that the ground state energy of H_1 is

$$E_B \equiv \langle\psi_1|H|\psi_1\rangle = -\frac{1}{2}K\lambda\,.$$

Now consider the Hamiltonian

$$H = \frac{p^2}{2m} - K\delta(x) - K\delta(x-R)\,,$$

and let $E_0(R)$ be its ground-state energy as a function of R. Assuming that

$$\psi_2(x) = \sqrt{\lambda}\, e^{-\lambda|x-R|}\,,$$

use $(*)$ to compute s, $\mathcal{E}$ and ϵ for ψ_1 and ψ_2 as given. Hence show that

$$E_0(R) \leqslant E_B \left[1 + 2\,\frac{e^{-\lambda R}\left(1 + e^{-\lambda R}\right)}{1 + (1+\lambda R)\, e^{-\lambda R}}\right].$$

Why should you expect this inequality to become an approximate equality for sufficiently large R? Describe briefly how this is relevant to molecular binding.

3/II/33B **Applications of Quantum Mechanics**

Let $\{\mathbf{l}\}$ be the set of lattice vectors of some lattice. Define the reciprocal lattice. What is meant by a Bravais lattice?

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be mutually orthogonal unit vectors. A crystal has identical atoms at positions given by the vectors

$$\begin{aligned} &a\big[n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}\big]\,, && a\big[(n_1 + \tfrac{1}{2})\mathbf{i} + (n_2 + \tfrac{1}{2})\mathbf{j} + n_3\mathbf{k}\big]\,, \\ &a\big[(n_1 + \tfrac{1}{2})\mathbf{i} + \mathbf{j} + (n_3 + \tfrac{1}{2})\mathbf{k}\big]\,, && a\big[n_1\mathbf{i} + (n_2 + \tfrac{1}{2})\mathbf{j} + (n_3 + \tfrac{1}{2})\mathbf{k}\big]\,, \end{aligned}$$

where (n_1, n_2, n_3) are arbitrary integers and a is a constant. Show that these vectors define a Bravais lattice with basis vectors

$$\mathbf{a}_1 = a\tfrac{1}{2}(\mathbf{j} + \mathbf{k})\,, \qquad \mathbf{a}_2 = a\tfrac{1}{2}(\mathbf{i} + \mathbf{k})\,, \qquad \mathbf{a}_3 = a\tfrac{1}{2}(\mathbf{i} + \mathbf{j})\,.$$

Verify that a basis for the reciprocal lattice is

$$\mathbf{b}_1 = \frac{2\pi}{a}(\mathbf{j} + \mathbf{k} - \mathbf{i})\,, \qquad \mathbf{b}_2 = \frac{2\pi}{a}(\mathbf{i} + \mathbf{k} - \mathbf{j})\,, \qquad \mathbf{b}_3 = \frac{2\pi}{a}(\mathbf{i} + \mathbf{j} - \mathbf{k})\,.$$

In Bragg scattering, an incoming plane wave of wave-vector $\mathbf{k}$ is scattered to an outgoing wave of wave-vector $\mathbf{k}'$. Explain why $\mathbf{k}' = \mathbf{k} + \mathbf{g}$ for some reciprocal lattice vector $\mathbf{g}$. Given that θ is the scattering angle, show that

$$\sin\frac{1}{2}\theta = \frac{|\mathbf{g}|}{2\,|\mathbf{k}|}\,.$$

For the above lattice, explain why you would expect scattering through angles θ_1 and θ_2 such that

$$\frac{\sin\frac{1}{2}\theta_1}{\sin\frac{1}{2}\theta_2} = \frac{\sqrt{3}}{2}\,.$$

4/II/33B **Applications of Quantum Mechanics**

A semiconductor has a valence energy band with energies $E \leqslant 0$ and density of states $g_v(E)$, and a conduction energy band with energies $E \geqslant E_g$ and density of states $g_c(E)$. Assume that $g_v(E) \sim A_v(-E)^{\frac{1}{2}}$ as $E \to 0$, and that $g_c(E) \sim A_c(E - E_g)^{\frac{1}{2}}$ as $E \to E_g$. At zero temperature all states in the valence band are occupied and the conduction band is empty. Let p be the number of holes in the valence band and n the number of electrons in the conduction band at temperature T. Under suitable approximations derive the result

$$pn = N_v N_c e^{-E_g/kT}$$

where

$$N_v = \tfrac{1}{2}\sqrt{\pi} A_v (kT)^{\frac{3}{2}}\,, \qquad N_c = \tfrac{1}{2}\sqrt{\pi} A_c (kT)^{\frac{3}{2}}\,.$$

Briefly describe how a semiconductor may conduct electricity but with a conductivity that is strongly temperature dependent.

Describe how doping of the semiconductor leads to $p \neq n$. A pn junction is formed between an n-type semiconductor, with N_d donor atoms, and a p-type semiconductor, with N_a acceptor atoms. Show that there is a potential difference $V_{np} = \Delta E/|e|$ across the junction, where e is the electron charge, and

$$\Delta E = E_g - kT \ln \frac{N_v N_c}{N_d N_a}\,.$$

Two semiconductors, one p-type and one n-type, are joined to make a closed circuit with two pn junctions. Explain why a current will flow around the circuit if the junctions are at different temperatures.

[*The Fermi–Dirac distribution function at temperature T and chemical potential μ is $\dfrac{g(E)}{e^{(E-\mu)/kT}+1}$, where $g(E)$ is the number of states with energy E.*

Note that $\displaystyle\int_0^\infty x^{\frac{1}{2}} e^{-x}\, dx = \tfrac{1}{2}\sqrt{\pi}$.]

2/II/34D **Statistical Physics**

Write down the first law of thermodynamics in differential form applied to an infinitesimal reversible change.

Explain what is meant by an adiabatic change.

Starting with the first law in differential form, derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V .$$

Hence show that

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P .$$

For radiation in thermal equilibrium at temperature T in volume V, it is given that $E = Ve(T)$ and $P = e(T)/3$. Hence deduce Stefan's Law,

$$E = aVT^4 ,$$

where a is a constant.

The radiation is allowed to expand adiabatically. Show that VT^3 is constant during the expansion.

3/II/34D **Statistical Physics**

A free spinless particle moving in two dimensions is confined to a square box of side L. By imposing periodic boundary conditions show that the number of states in the energy range $\epsilon \to \epsilon + d\epsilon$ is $g(\epsilon)d\epsilon$, where

$$g(\epsilon) = \frac{mL^2}{2\pi\hbar^2} .$$

If, instead, the particle is an electron with magnetic moment μ moving in a constant external magnetic field H, show that

$$g(\epsilon) = \begin{cases} \dfrac{mL^2}{2\pi\hbar^2}, & -\mu H < \epsilon < \mu H \\ \dfrac{mL^2}{\pi\hbar^2}, & \mu H < \epsilon . \end{cases}$$

Let there be N electrons in the box. Explain briefly how to construct the ground state of the system. Let ϵ be the Fermi energy. Show that when $\epsilon > \mu H$

$$N = \frac{mL^2}{\pi\hbar^2}\epsilon .$$

Show also that the magnetic moment M of the system in its ground state is given by

$$M = \frac{\mu^2 mL^2}{\pi\hbar^2} H ,$$

and that the ground state energy is

$$\frac{1}{2}\frac{\pi\hbar^2}{mL^2}N^2 - \frac{1}{2}MH .$$

4/II/34D **Statistical Physics**

Write down an expression for the partition function of a classical particle of mass m moving in three dimensions in a potential $U(\mathbf{x})$ and in equilibrium with a heat bath at temperature T.

A system of N non-interacting classical particles is placed in the potential

$$U(\mathbf{x}) \;=\; \frac{(x^2+y^2+z^2)^n}{V^{2n/3}}\;,$$

where n is a positive integer. The gas is in equilibrium at temperature T. Using a suitable rescaling of variables, show that the free energy F is given by

$$\frac{F}{N} \;=\; -kT\left(\log V + \frac{3}{2}\frac{n+1}{n}\log kT + \log I_n\right)\;,$$

where

$$I_n = \left(\frac{2m\pi}{h^2}\right)^{3/2}\int_0^\infty 4\pi u^2 e^{-u^{2n}}\,du\,.$$

Regarding V as an external parameter, find the thermodynamic force P, conjugate to V, exerted by this system. Find the equation of state and compare with that of an ideal gas confined in a volume V.

Derive expressions for the entropy S, the internal energy E and the total heat capacity C_V at constant V.

Show that for all n the total heat capacity at constant P is given by

$$C_P \;=\; C_V + Nk\;.$$

[*Note that* $\int_0^\infty u^2 e^{-u^2/2}\,du \;=\; \sqrt{\frac{\pi}{2}}\;.$]

1/II/34B **Electrodynamics**

In a frame $\mathcal{F}$ the electromagnetic fields $(\mathbf{E}, \mathbf{B})$ are encoded into the Maxwell field 4-tensor F^{ab} and its dual ${}^*F^{ab}$, where

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and

$${}^*F^{ab} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} .$$

[*Here the signature is* $(+ \; - \; - \; -)$ *and units are chosen so that* $c = 1$.] Obtain two independent Lorentz scalars of the electromagnetic field in terms of $\mathbf{E}$ and $\mathbf{B}$.

Suppose that $\mathbf{E}\cdot\mathbf{B} > 0$ in the frame $\mathcal{F}$. Given that there exists a frame $\mathcal{F}'$ in which $\mathbf{E}' \times \mathbf{B}' = \mathbf{0}$, show that

$$E' = \left[(\mathbf{E}\cdot\mathbf{B}) \big(K + \sqrt{1+K^2} \big) \right]^{1/2} , \qquad B' = \left[\frac{\mathbf{E}\cdot\mathbf{B}}{K + \sqrt{1+K^2}} \right]^{1/2} ,$$

where (E', B') are the magnitudes of $(\mathbf{E}', \mathbf{B}')$, and

$$K = \frac{1}{2} \big(|\mathbf{E}|^2 - |\mathbf{B}|^2 \big) / \big(\mathbf{E}\cdot\mathbf{B} \big) .$$

[*Hint: there is no need to consider the Lorentz transformations for* $\mathbf{E}'$ *and* $\mathbf{B}'$.]

3/II/35B **Electrodynamics**

A non-relativistic particle of rest mass m and charge q is moving slowly with velocity $\mathbf{v}(t)$. The power $dP/d\Omega$ radiated per unit solid angle in the direction of a unit vector $\mathbf{n}$ is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2} \left|\mathbf{n} \times q\dot{\mathbf{v}}\right|^2 .$$

Obtain Larmor's formula

$$P = \frac{\mu_0 \, q^2}{6\pi} \left|\dot{\mathbf{v}}\right|^2 .$$

The particle has energy $\mathcal{E}$ and, starting from afar, makes a head-on collision with a fixed central force described by a potential $V(r)$, where $V(r) > \mathcal{E}$ for $r < r_0$ and $V(r) < \mathcal{E}$ for $r > r_0$. Let W be the total energy radiated by the particle. Given that $W \ll \mathcal{E}$, show that

$$W \approx \frac{\mu_0 \, q^2}{3\pi m^2} \sqrt{\frac{m}{2}} \int_{r_0}^{\infty} \left(\frac{dV}{dr}\right)^2 \frac{dr}{\sqrt{V(r_0) - V(r)}} .$$

4/II/35B **Electrodynamics**

In Ginzburg–Landau theory, superconductivity is due to "supercarriers" of mass m_s and charge q_s, which are described by a macroscopic wavefunction ψ with "Mexican hat" potential

$$V = \alpha(T)|\psi|^2 + \frac{1}{2}\beta|\psi|^4 .$$

Here, $\beta > 0$ is constant and $\alpha(T)$ is a function of temperature T such that $\alpha(T) > 0$ for $T > T_c$ but $\alpha(T) < 0$ for $T < T_c$, where T_c is a critical temperature. In the presence of a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, the total energy of the superconducting system is

$$E[\psi, \psi^*, \mathbf{A}] = \int d^3x \left[\frac{1}{2\mu_0} A_{k,j}\big(A_{k,j} - A_{j,k}\big) + \frac{\hbar^2}{2m_s}\left|\psi_{,k} + i\frac{q_s}{\hbar}A_k\psi\right|^2 + V \right].$$

Use this to derive the equations

$$-\frac{\hbar^2}{2m_s}\big(\boldsymbol{\nabla} - i\frac{q_s}{\hbar}\mathbf{A}\big)^2\psi + \big(\alpha + \beta|\psi|^2\big)\psi = 0 \qquad (*)$$

and

$$\boldsymbol{\nabla} \times \mathbf{B} \equiv \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\mathbf{A}\big) - \boldsymbol{\nabla}^2\mathbf{A} = \mu_0\,\mathbf{j}\,, \qquad (\dagger)$$

where

$$\begin{aligned} \mathbf{j} &= -\frac{iq_s\hbar}{2m_s}\big(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\big) - \frac{q_s^2}{m_s}|\psi|^2\mathbf{A} \\ &= \frac{q_s}{2m_s}\left[\psi^*\left(-i\hbar\boldsymbol{\nabla} - q_s\mathbf{A}\right)\psi + \psi\left(i\hbar\boldsymbol{\nabla} - q_s\mathbf{A}\right)\psi^*\right] . \end{aligned}$$

Suppose that we write the wavefunction as

$$\psi = \sqrt{n_s}\, e^{i\theta}\,,$$

where n_s is the (real positive) supercarrier density and θ is a real phase function. Given that

$$\left(\boldsymbol{\nabla} - \frac{iq_s}{\hbar}\,\mathbf{A}\right)\psi = 0\,,$$

show that n_s is constant and that $\hbar\boldsymbol{\nabla}\theta = q_s\mathbf{A}$. Given also that $T < T_c$, deduce that $(*)$ allows such solutions for a certain choice of n_s, which should be determined. Verify that your results imply $\mathbf{j} = \mathbf{0}$. Show also that $\mathbf{B} = \mathbf{0}$ and hence that $(\dagger)$ is solved.

Let $\mathcal{S}$ be a surface within the superconductor with closed boundary $\mathcal{C}$. Show that the magnetic flux through $\mathcal{S}$ is

$$\Phi \equiv \int_{\mathcal{S}} \mathbf{B}\cdot\mathbf{dS} = \frac{\hbar}{q_s}\big[\theta\big]_{\mathcal{C}}\,.$$

Discuss, briefly, flux quantization.

1/II/35C **General Relativity**

Suppose $(x(\tau), t(\tau))$ is a timelike geodesic of the metric

$$ds^2 = \frac{dx^2}{1+x^2} - (1+x^2)\, dt^2 \,,$$

where τ is proper time along the world line. Show that $dt/d\tau = E/(1+x^2)$, where $E > 1$ is a constant whose physical significance should be stated. Setting $a^2 = E^2 - 1$, show that

$$d\tau = \frac{dx}{\sqrt{a^2 - x^2}} \,, \qquad dt = \frac{E\; dx}{(1+x^2)\sqrt{a^2-x^2}} \,. \tag{$*$}$$

Deduce that x is a periodic function of proper time τ with period 2π. Sketch $dx/d\tau$ as a function of x and superpose on this a sketch of dx/dt as a function of x. Given the identity

$$\int_{-a}^{a} \frac{E\, dx}{(1+x^2)\sqrt{a^2-x^2}} = \pi \,,$$

deduce that x is also a periodic function of t with period 2π.

Next consider the family of metrics

$$ds^2 = \frac{[1+f(x)]^2\, dx^2}{1+x^2} - (1+x^2)\, dt^2 \,,$$

where f is an odd function of x, $f(-x) = -f(x)$, and $|f(x)| < 1$ for all x. Derive expressions analogous to $(*)$ above. Deduce that x is a periodic function of τ and also that x is a periodic function of t. What are the periods?

2/II/35C **General Relativity**

State without proof the properties of local inertial coordinates x^a centred on an arbitrary spacetime event p. Explain their physical significance.

Obtain an expression for $\partial_a \Gamma_b{}^c{}_d$ at p in inertial coordinates. Use it to derive the formula

$$R_{abcd} = \tfrac{1}{2}\big(\partial_b\partial_c g_{ad} + \partial_a\partial_d g_{bc} - \partial_b\partial_d g_{ac} - \partial_a\partial_c g_{bd}\big)$$

for the components of the Riemann tensor at p in local inertial coordinates. Hence deduce that at any point in any chart $R_{abcd} = R_{cdab}$.

Consider the metric

$$ds^2 = \frac{\eta_{ab}\, dx^a\, dx^b}{(1 + L^{-2}\eta_{ab}x^a x^b)^2} \,,$$

where $\eta_{ab} = \mathrm{diag}(1,1,1,-1)$ is the Minkowski metric tensor and L is a constant. Compute the Ricci scalar $R = R^{ab}{}_{ab}$ at the origin $x^a = 0$.

4/II/36C **General Relativity**

State clearly, but do not prove, Birkhoff's Theorem about spherically symmetric spacetimes. Let (r, θ, ϕ) be standard spherical polar coordinates and define $F(r) = 1 - 2M/r$, where M is a constant. Consider the metric

$$ds^2 = \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - F(r)\, dt^2.$$

Explain carefully why this is appropriate for the region outside a spherically symmetric star that is collapsing to form a black hole.

By considering radially infalling timelike geodesics $x^a = (r(\tau), 0, 0, t(\tau))$, where τ is proper time along the curve, show that a freely falling observer will reach the event horizon after a finite proper time. Show also that a distant observer would see the horizon crossing only after an infinite time.

1/II/36E Fluid Dynamics II

Consider a unidirectional flow with dynamic viscosity μ along a straight rigid-walled channel of uniform cross-sectional shape $\mathcal{D}$ driven by a uniform applied pressure gradient G. Write down the differential equation and boundary conditions governing the velocity w along the channel.

Consider the situation when the boundary includes a sharp corner of angle 2α. Explain why one might expect that, sufficiently close to the corner, the solution should be of the form

$$w = (G/\mu) r^2 f(\theta)\,,$$

where r and θ are polar co-ordinates with origin at the vertex and $\theta = \pm\alpha$ describing the two planes emanating from the corner. Determine $f(\theta)$.

If $\mathcal{D}$ is the sector bounded by the lines $\theta = \pm\alpha$ and the circular arc $r = a$, show that the flow is given by

$$w = (G/\mu) r^2 f(\theta) + \sum_{n=0}^{\infty} A_n r^{\lambda_n} \cos \lambda_n \theta,$$

where λ_n and A_n are to be determined.

[*Note that* $\int \cos(ax)\cos(bx)\,dx = \{a\sin(ax)\cos(bx) - b\sin(bx)\cos(ax)\}/(a^2 - b^2)$.]

Considering the values of λ_0 and λ_1, comment briefly on the cases: (i) $2\alpha < \frac{1}{2}\pi$; (ii) $\frac{1}{2}\pi < 2\alpha < \frac{3}{2}\pi$; and (iii) $\frac{3}{2}\pi < 2\alpha < 2\pi$.

2/II/36E Fluid Dynamics II

A volume V of very viscous fluid of density ρ and dynamic viscosity μ is released at the origin on a rigid horizontal boundary at time $t = 0$. Using lubrication theory, determine the velocity profile in the gravity current once it has spread sufficiently that the axisymmetric thickness $h(r,t)$ of the current is much less than the radius $R(t)$ of the front.

Derive the differential equation

$$\frac{\partial h}{\partial t} = \frac{\beta}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right)\,,$$

where β is to be determined.

Write down the other equations that are needed to determine the appropriate similarity solution for this problem.

Determine the similarity solution and calculate $R(t)$.

3/II/36E **Fluid Dynamics II**

Write down the Navier–Stokes equations for an incompressible fluid.

Explain the concepts of the Euler and Prandtl limits applied to the Navier–Stokes equations near a rigid boundary.

A steady two-dimensional flow given by $(U, 0)$ far upstream flows past a semi-infinite flat plate, held at $y = 0, x > 0$. Derive the boundary layer equation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

for the stream-function $\psi(x, y)$ near the plate, explaining any approximations made.

Show that the appropriate solution must be of the form

$$\psi(x, y) = (\nu U x)^{1/2} f(\eta),$$

and determine the dimensionless variable η.

Derive the equation and boundary conditions satisfied by $f(\eta)$. [*You need not solve them.*]

Suppose now that the plate has a *finite* length L in the direction of the flow. Show that the force F on the plate (per unit width perpendicular to the flow) is given by

$$F = \frac{4\rho U^2 L}{(UL/\nu)^{1/2}} \frac{f''(0)}{[f'(\infty)]^2}.$$

4/II/37E **Fluid Dynamics II**

Consider flow of an incompressible fluid of uniform density ρ and dynamic viscosity μ. Show that the rate of viscous dissipation per unit volume is given by

$$\Phi = 2\mu e_{ij} e_{ij} \,,$$

where e_{ij} is the strain rate.

Determine expressions for e_{ij} and Φ when the flow is irrotational with velocity potential ϕ. Hence determine the rate of viscous dissipation, averaged over a wave period $2\pi/\omega$, for an irrotational two-dimensional surface wave of wavenumber k and small amplitude $a \ll k^{-1}$ in a fluid of very small viscosity $\mu \ll \rho\omega/k^2$ and great depth $H \gg 1/k$.

[*You may use without derivation that in deep water a linearised wave with surface displacement* $\eta = a\cos\,(kx - wt)$ *has velocity potential* $\phi = -(\omega a/k)e^{-kz}\sin\,(kx - \omega t)$.]

Calculate the depth-integrated time-averaged kinetic energy per wavelength. Assuming that the average potential energy is equal to the average kinetic energy, show that the total wave energy decreases to leading order like $e^{-\gamma t}$, where

$$\gamma = 4\mu k^2/\rho \,.$$

1/II/37E Waves

An elastic solid with density ρ has Lamé moduli λ and μ. Write down equations satisfied by the dilational and shear potentials ϕ and $\boldsymbol{\psi}$.

For a two-dimensional disturbance give expressions for the displacement field $\mathbf{u} = (u_x, u_y, 0)$ in terms of $\phi(x, y; t)$ and $\boldsymbol{\psi} = (0, 0, \psi(x, y; t))$.

Suppose the solid occupies the region $y < 0$ and that the surface $y = 0$ is free of traction. Find a combination of solutions for ϕ and ψ that represent a propagating surface wave (a Rayleigh wave) near $y = 0$. Show that the wave is non-dispersive and obtain an equation for the speed c. [*You may assume without proof that this equation has a unique positive root.*]

2/II/37E Waves

Show that, in the standard notation for a one-dimensional flow of a perfect gas at constant entropy, the quantity $u + 2(c - c_0)/(\gamma - 1)$ remains constant along characteristics $dx/dt = u + c$.

A perfect gas is initially at rest and occupies a tube in $x > 0$. A piston is pushed into the gas so that its position at time t is $x(t) = \frac{1}{2} f t^2$, where $f > 0$ is a constant. Find the time and position at which a shock first forms in the gas.

3/II/37E Waves

The real function $\phi(x, t)$ satisfies the equation

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = \frac{\partial^3 \phi}{\partial x^3},$$

where $U > 0$ is a constant. Find the dispersion relation for waves of wavenumber k and deduce whether wave crests move faster or slower than a wave packet.

Suppose that $\phi(x, 0)$ is given by a Fourier transform as

$$\phi(x, 0) = \int_{-\infty}^{\infty} A(k) e^{ikx} dk.$$

Use the method of stationary phase to find $\phi(Vt, t)$ as $t \to \infty$ for fixed $V > U$.

[*You may use the result that* $\int_{-\infty}^{\infty} e^{-a\xi^2} d\xi = (\pi/a)^{1/2}$ *if* $\mathrm{Re}(a) \geqslant 0$.]

What can be said if $V < U$? [*Detailed calculation is* **not** *required in this case.*]

4/II/38E **Waves**

Starting from the equations of conservation of mass and momentum for an inviscid compressible fluid, show that for small perturbations about a state of rest and uniform density the velocity is irrotational and the velocity potential satisfies the wave equation. Identify the sound speed c_0.

Define the acoustic energy density and acoustic energy flux, and derive the equation for conservation of acoustic energy.

Show that in any (not necessarily harmonic) acoustic plane wave of wavenumber $\mathbf{k}$ the kinetic and potential energy densities are equal and that the acoustic energy is transported with velocity $c_0\widehat{\mathbf{k}}$.

Calculate the kinetic and potential energy densities for a spherically symmetric outgoing wave. Are they equal?

1/II/38A **Numerical Analysis**

Let

$$\frac{\mu}{4}u_{m-1}^{n+1} + u_m^{n+1} - \frac{\mu}{4}u_{m+1}^{n+1} = -\frac{\mu}{4}u_{m-1}^{n} + u_m^{n} + \frac{\mu}{4}u_{m+1}^{n},$$

where n is a positive integer and m ranges over all integers, be a finite-difference method for the advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad -\infty < x < \infty, \quad t \geqslant 0.$$

Here $\mu = \frac{\Delta t}{\Delta x}$ is the Courant number.

(a) Show that the local error of the method is $O((\Delta x)^3)$.

(b) Determine the range of $\mu > 0$ for which the method is stable.

2/II/38A **Numerical Analysis**

Define a *Krylov subspace* $\mathcal{K}_n(A, v)$.

Let d_n be the dimension of $\mathcal{K}_n(A, v)$. Prove that the sequence $\{d_m\}_{m=1,2,\ldots}$ increases monotonically. Show that, moreover, there exists an integer k with the following property: $d_m = m$ for $m = 1, 2, \ldots, k$, while $d_m = k$ for $m \geqslant k$. Assuming that A has a full set of eigenvectors, show that k is equal to the number of eigenvectors of A required to represent the vector v.

3/II/38A **Numerical Analysis**

Consider the Runge–Kutta method

$$\begin{aligned} k_1 &= f(y_n), \\ k_2 &= f(y_n + (1-a)hk_1 + ahk_2), \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2) \end{aligned}$$

for the solution of the scalar ordinary differential equation $y' = f(y)$. Here a is a real parameter.

(a) Determine the order of the method.

(b) Find the range of values of a for which the method is A-stable.

4/II/39A Numerical Analysis

An $n \times n$ *skew-symmetric* matrix A is converted into an upper-Hessenberg form B, say, by Householder reflections.

(a) Describe each step of the procedure and observe that B is tridiagonal. Your algorithm should take advantage of the special form of A to reduce the number of computations.

(b) Compare the cost (counting only products and looking only at the leading term) of converting a skew-symmetric and a symmetric matrix to an upper-Hessenberg form using Householder reflections.