UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday 10 June 2005 9 to 12

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***,...,***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

If n is an odd integer and b is an integer prime with n, state what it means for n to be a pseudoprime to the base b. What is a Carmichael number? State a criterion for n to be a Carmichael number and use the criterion to show that:

(i) Every Carmichael number is the product of at least three distinct primes.

(ii) 561 is a Carmichael number.

2F Topics in Analysis

(i) Let $D \subset \mathbb{C}$ be a domain, let $f : D \to \mathbb{C}$ be an analytic function and let $z_0 \in D$. What does Taylor's theorem say about z_0, f and D?

(ii) Let K be the square consisting of all complex numbers z such that

$$-1 \leqslant \operatorname{Re}(z) \leqslant 1 \text{ and } -1 \leqslant \operatorname{Im}(z) \leqslant 1\,,$$

and let w be a complex number not belonging to K. Prove that the function $f(z) = (z-w)^{-1}$ can be uniformly approximated on K by polynomials.

3G Geometry of Group Actions

Show that a set $F \subset \mathbb{R}^n$ with Hausdorff dimension strictly less than one is totally disconnected.

What does it mean for a Möbius transformation to *pair two discs*? By considering a pair of disjoint discs and a pair of tangent discs, or otherwise, explain in words why there is a 2-generator Schottky group with limit set $\Lambda \subset \mathbb{S}^2$ which has Hausdorff dimension at least 1 but which is not homeomorphic to a circle.

4J Coding and Cryptography

What does it mean to transmit reliably at rate r through a binary symmetric channel (BSC) with error probability p? Assuming Shannon's second coding theorem, compute the supremum of all possible reliable transmission rates of a BSC. What happens if (i) p is very small, (ii) $p = 1/2$, or (iii) $p > 1/2$?

5I Statistical Modelling

You see below five R commands, and the corresponding output (which is slightly abbreviated). Without giving any mathematical proofs, explain the purpose of these commands, and interpret the output.

```
> Yes = c(12, 27,11,24)

> Total = c(117,170,52,118)

> Sclass = c("a","a","b","b")

> Sclass = factor(Sclass)

> summary(glm(Yes/Total~ Sclass, binomial, weights=Total))

Coefficients:

            Estimate Std. Error z value

(Intercept)  -1.8499     0.1723 -10.739

Sclassb       0.4999     0.2562   1.951


Residual deviance: 1.9369  on 2  degrees of freedom


Number of Fisher Scoring iterations: 4
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6E Mathematical Biology

The output of a linear perceptron is given by $y = \mathbf{w} \cdot \mathbf{x}$, where $\mathbf{w}$ is a vector of weights connecting a fluctuating input vector $\mathbf{x}$ to an output unit. The weights are given random initial values and are then updated according to a learning rule that has a time-constant τ much greater than the fluctuation timescale of the inputs.

(a) Find the behaviour of $|\mathbf{w}|$ for each of the following two rules

(i) $\tau \dfrac{d\mathbf{w}}{dt} = y\mathbf{x}$

(ii) $\tau \dfrac{d\mathbf{w}}{dt} = y\mathbf{x} - \alpha y^2 \mathbf{w}|\mathbf{w}|^2$, where α is a positive constant.

(b) Consider a third learning rule

(iii) $\tau \dfrac{d\mathbf{w}}{dt} = y\mathbf{x} - \mathbf{w}|\mathbf{w}|^2 \,.$

Show that in a steady state the vector of weights satisfies the eigenvalue equation

$$\mathbf{C}\mathbf{w} = \lambda \mathbf{w} \,,$$

where the matrix $\mathbf{C}$ and eigenvalue λ should be identified.

(c) Comment briefly on the biological implications of the three rules.

7B Dynamical Systems

Find and classify the fixed points of the system

$$\begin{aligned} \dot{x} &= x(1-y) \\ \dot{y} &= -y + x^2 \,. \end{aligned}$$

Sketch the phase plane.

What is the ω-limit for the point $(2,-1)$? Which points have $(0,0)$ as their ω-limit?

8A Further Complex Methods

Write down necessary and sufficient conditions on the functions $p(z)$ and $q(z)$ for the point $z=0$ to be (i) an ordinary point and (ii) a regular singular point of the equation

$$w'' + p(z)w' + q(z)w = 0. \qquad (*)$$

Show that the point $z=\infty$ is an ordinary point if and only if

$$p(z) = 2z^{-1} + z^{-2}P(z^{-1}), \qquad q(z) = z^{-4}Q(z^{-1}),$$

where P and Q are analytic in a neighbourhood of the origin.

Find the most general equation of the form $(*)$ that has a regular singular point at $z=0$ but no other singular points.

9C Classical Dynamics

Define a canonical transformation for a one-dimensional system with coordinates $(q,p) \to (Q,P)$. Show that if the transformation is canonical then $\{Q,P\}=1$.

Find the values of constants α and β such that the following transformations are canonical:

(i) $Q = pq^{\beta},\ P = \alpha q^{-1}$.

(ii) $Q = q^{\alpha}\cos(\beta p),\ P = q^{\alpha}\sin(\beta p)$.

10D Cosmology

The linearised equation for the growth of a density fluctuation δ_k in a homogeneous and isotropic universe is

$$\frac{d^2\delta_k}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta_k}{dt} - \left(4\pi G\rho_{\rm m} - \frac{v_s^2 k^2}{a^2}\right)\delta_k = 0\,, \qquad (*)$$

where $\rho_{\rm m}$ is the non-relativistic matter density, k is the comoving wavenumber and v_s is the sound speed ($v_s^2 \equiv dP/d\rho$).

(a) Define the Jeans length $\lambda_{\rm J}$ and discuss its significance for perturbation growth.

(b) Consider an Einstein–de Sitter universe with $a(t) = (t/t_0)^{2/3}$ filled with pressure-free matter ($P=0$). Show that the perturbation equation $(*)$ can be re-expressed as

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0\,.$$

By seeking power law solutions, find the growing and decaying modes of this equation.

(c) Qualitatively describe the evolution of non-relativistic matter perturbations ($k > aH$) in the radiation era, $a(t) \propto t^{1/2}$, when $\rho_{\rm r} \gg \rho_{\rm m}$. What feature in the power spectrum is associated with the matter–radiation transition?

SECTION II

11H Number Theory

(a) Let N be a non-square integer. Describe the integer solutions of the Pell equation $x^2 - Ny^2 = 1$ in terms of the convergents to $\sqrt{N}$. Show that the set of integer solutions forms an abelian group. Denote the addition law in this group by $\circ$; given solutions (x_0, y_0) and (x_1, y_1), write down an explicit formula for $(x_0, y_0) \circ (x_1, y_1)$. If (x, y) is a solution, write down an explicit formula for $(x, y) \circ (x, y) \circ (x, y)$ in the group law.

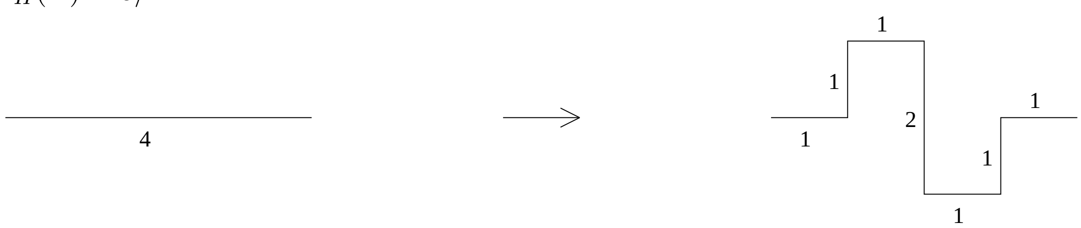
(b) Find the continued fraction expansion of $\sqrt{11}$. Find the smallest solution in integers $x, y > 0$ of the Pell equation $x^2 - 11y^2 = 1$. Use the formula in Part (a) to compute $(x, y) \circ (x, y) \circ (x, y)$.

12G Geometry of Group Actions

For real $s \geqslant 0$ and $F \subset \mathbb{R}^n$, give a careful definition of the s-dimensional *Hausdorff measure* of F and of the *Hausdorff dimension* $\dim_H(F)$ of F.

For $1 \leqslant i \leqslant k$, suppose $S_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with contraction factor $c_i \in (0, 1)$. Prove there is a unique non-empty compact invariant set I for the $\{S_i\}$. State a formula for the Hausdorff dimension of I, under an assumption on the S_i you should state.

Hence show the Hausdorff dimension of the fractal F given by iterating the scheme below (at each stage replacing each edge by a new copy of the generating template) is $\dim_H(F) = 3/2$.



[Numbers denote lengths]

13I Statistical Modelling

(i) Suppose that $Y_1, \ldots, Y_n$ are independent random variables, and that Y_i has probability density function

$$f(y_i|\beta,\nu) = \left(\frac{\nu y_i}{\mu_i}\right)^{\nu} e^{-y_i\nu/\mu_i} \frac{1}{\Gamma(\nu)} \frac{1}{y_i} \quad \text{for } y_i > 0$$

where

$$1/\mu_i = \beta^T x_i \ , \text{ for } \quad 1 \leqslant i \leqslant n,$$

and $x_1, \ldots, x_n$ are given p-dimensional vectors, and ν is known.

Show that $\mathbb{E}(Y_i) = \mu_i$ and that $\operatorname{var}(Y_i) = \mu_i^2/\nu$.

(ii) Find the equation for $\hat{\beta}$, the maximum likelihood estimator of β, and suggest an iterative scheme for its solution.

(iii) If $p = 2$, and $x_i = \begin{pmatrix} 1 \\ z_i \end{pmatrix}$, find the large-sample distribution of $\hat{\beta}_2$. Write your answer in terms of a, b, c and ν, where a, b, c are defined by

$$a = \sum \mu_i^2, \quad b = \sum z_i \mu_i^2, \quad c = \sum z_i^2 \mu_i^2 .$$

14A Further Complex Methods

Two representations of the zeta function are

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1} dt \qquad \text{and} \qquad \zeta(z) = \sum_{1}^{\infty} n^{-z} \ ,$$

where, in the integral representation, the path is the Hankel contour and the principal branch of t^{z-1}, for which $|\arg z| < \pi$, is to be used. State the range of z for which each representation is valid.

Evaluate the integral

$$\int_{\gamma} \frac{t^{z-1}}{e^{-t}-1} dt,$$

where γ is a closed path consisting of the straight line $z = \pi i + x$, with $|x| < 2N\pi$, and the semicircle $|z - \pi i| = 2N\pi$, with $\operatorname{Im} z > \pi$, where N is a positive integer.

Making use of this result and assuming, when necessary, that the integral along the curved part of γ is negligible when N is large, derive the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z)$$

for $z \neq 1$.

15D Cosmology

For an ideal gas of *bosons*, the average occupation number can be expressed as

$$\bar{n}_k = \frac{g_k}{e^{(E_k-\mu)/kT} - 1}\,, \qquad (*)$$

where g_k has been included to account for the degeneracy of the energy level E_k. In the approximation in which a discrete set of energies E_k is replaced with a continuous set with momentum p, the density of one-particle states with momentum in the range p to $p + dp$ is $g(p)dp$. Explain briefly why

$$g(p) \;\propto\; p^2 V\,,$$

where V is the volume of the gas. Using this formula with equation $(*)$, obtain an expression for the total energy density $\epsilon = E/V$ of an ultra-relativistic gas of bosons at zero chemical potential as an integral over p. Hence show that

$$\epsilon \;\propto\; T^\alpha\,,$$

where α is a number you should find. Why does this formula apply to photons?

Prior to a time $t \sim 100{,}000$ years, the universe was filled with a gas of photons and non-relativistic free electrons and protons. Subsequently, at around $t \sim 400{,}000$ years, the protons and electrons began combining to form neutral hydrogen,

$$p + e^- \leftrightarrow H + \gamma\,.$$

Deduce Saha's equation for this recombination process stating clearly the steps required:

$$\frac{n_{\rm e}^2}{n_{\rm H}} = \left(\frac{2\pi m_{\rm e} kT}{h^2}\right)^{3/2} \exp(-I/kT)\,,$$

where I is the ionization energy of hydrogen. [*Note that the equilibrium number density of a non-relativistic species* ($kT \ll mc^2$) *is given by* $n = g_s \left(\frac{2\pi m kT}{h^2}\right)^{3/2} \exp\left[(\mu - mc^2)/kT\right]$, *while the photon number density is* $n_\gamma = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3$, where $\zeta(3) \approx 1.20....$]

Consider now the fractional ionization $X_{\rm e} = n_{\rm e}/n_{\rm B}$, where $n_B = n_{\rm p} + n_{\rm H} = \eta n_\gamma$ is the baryon number of the universe and η is the baryon-to-photon ratio. Find an expression for the ratio

$$(1 - X_{\rm e})/X_{\rm e}^2$$

in terms only of kT and constants such as η and I. One might expect neutral hydrogen to form at a temperature given by $kT \approx I \approx 13\,$eV, but instead in our universe it forms at the much lower temperature $kT \approx 0.3\,$eV. Briefly explain why.

16F Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic. [*You do* **not** *need to give definitions of the various terms involved. You may assume that the set of primitive propositions is countable. You may also assume the Deduction Theorem, provided that you state it clearly.*]

Where in your argument have you used the third axiom, namely $(\neg\neg p) \Rightarrow p$?

State the Compactness Theorem, and deduce it from the Completeness Theorem.

17F Graph Theory

Write an essay on extremal graph theory. Your essay should include the proof of at least one extremal theorem. You should state the Erdős–Stone theorem, as well as describing its proof and showing how it can be applied.

18G Galois Theory

(i) Let K be the splitting field of the polynomial

$$x^4 - 4x^2 - 1$$

over $\mathbb{Q}$. Show that $[K : \mathbb{Q}] = 8$, and hence show that the Galois group of $K/\mathbb{Q}$ is the dihedral group of order 8.

(ii) Let L be the splitting field of the polynomial

$$x^4 - 4x^2 + 1$$

over $\mathbb{Q}$. Show that $[L : \mathbb{Q}] = 4$. Show that the Galois group of $L/\mathbb{Q}$ is $C_2 \times C_2$.

19G Representation Theory

(i) State and prove the Weyl integration formula for $SU(2)$.

(ii) Determine the characters of the symmetric powers of the standard 2-dimensional representation of $SU(2)$ and prove that they are irreducible.

[*Any general theorems from the course may be used.*]

20G Number Fields

State Dedekind's theorem on the factorisation of rational primes into prime ideals.

A rational prime is said to ramify totally in a field with degree n if it is the n-th power of a prime ideal in the field. Show that, in the quadratic field $\mathbb{Q}(\sqrt{d})$ with d a square-free integer, a rational prime ramifies totally if and only if it divides the discriminant of the field.

Verify that the same holds in the cyclotomic field $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/q}$ with q an odd prime, and also in the cubic field $\mathbb{Q}(\sqrt[3]{2})$.

[*The cases* $d \equiv 2, 3 \pmod 4$ *and* $d \equiv 1 \pmod 4$ *for the quadratic field should be carefully distinguished. It can be assumed that* $\mathbb{Q}(\zeta)$ *has a basis* $1, \zeta, \ldots, \zeta^{q-2}$ *and discriminant* $(-1)^{(q-1)/2} q^{q-1}$*, and that* $\mathbb{Q}(\sqrt[3]{2})$ *has a basis* $1, \sqrt[3]{2}, (\sqrt[3]{2})^2$ *and discriminant* -108.]

21H Algebraic Topology

Let X be a simplicial complex. Suppose $X = B \cup C$ for subcomplexes B and C, and let $A = B \cap C$. Show that the inclusion of A in B induces an isomorphism $H_* A \to H_* B$ if and only if the inclusion of C in X induces an isomorphism $H_* C \to H_* X$.

22F Linear Analysis

Let X and Y be normed vector spaces. Show that a linear map $T : X \to Y$ is continuous if and only if it is bounded.

Now let X, Y, Z be Banach spaces. We say that a map $F : X \times Y \to Z$ is *bilinear* if

$$F(\alpha x + \beta y, z) = \alpha F(x, z) + \beta F(y, z), \text{ for all scalars } \alpha, \beta \text{ and } x, y \in X, z \in Y$$

$$F(x, \alpha y + \beta z) = \alpha F(x, y) + \beta F(x, z), \text{ for all scalars } \alpha, \beta \text{ and } x \in X, y, z \in Y\,.$$

Suppose that F is bilinear and is continuous in each variable separately. Show that there exists a constant $M \geqslant 0$ such that

$$||F(x, y)|| \leqslant M ||x||\ ||y||$$

for all $x \in X$, $y \in Y$.

[*Hint: For each fixed* $x \in X$*, consider the map* $y \mapsto F(x, y)$ *from* Y *to* Z.]

23H Riemann Surfaces

Define what is meant by the *degree* of a non-constant holomorphic map between compact connected Riemann surfaces, and state the Riemann–Hurwitz formula.

Let $E_\Lambda = \mathbb{C}/\Lambda$ be an elliptic curve defined by some lattice Λ. Show that the map

$$\psi : z + \Lambda \in E_\Lambda \to -z + \Lambda \in E_\Lambda$$

is biholomorphic, and that there are four points in E_Λ fixed by ψ.

Let $S = E_\Lambda/\sim$ be the quotient surface (the topological surface obtained by identifying $z + \Lambda$ and $\psi(z + \Lambda)$, for each z) and let $\pi : E_\Lambda \to S$ be the corresponding projection map. Denote by $E^0_\Lambda \subset E_\Lambda$ the complement of the four points fixed by ψ, and let $S^0 = \pi(E^0_\Lambda)$. Describe briefly a family of charts making S^0 into a Riemann surface, so that $\pi : E^0_\Lambda \to S^0$ is a holomorphic map.

Now assume that the complex structure of S^0 extends to S, so that S is a Riemann surface, and that the map π is in fact holomorphic on all of E_Λ. Calculate the genus of S.

24H Differential Geometry

(i) Define what is meant by an *isothermal parametrization*. Let $\phi : U \to \mathbb{R}^3$ be an isothermal parametrization. Prove that

$$\phi_{uu} + \phi_{vv} = 2\,\lambda^2\,\mathbf{H},$$

where $\mathbf{H}$ is the mean curvature vector and $\lambda^2 = \langle \phi_u, \phi_u \rangle$.

Define what it means for ϕ to be *minimal*, and deduce that ϕ is minimal if and only if $\Delta\phi = 0$.

[*You may assume that the mean curvature H can be written as*

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}.\,]$$

(ii) Write $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$. Consider the complex valued functions

$$\varphi_1 = x_u - ix_v, \quad \varphi_2 = y_u - iy_v, \quad \varphi_3 = z_u - iz_v.$$

Show that ϕ is isothermal if and only if $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0$.

Suppose now that ϕ is isothermal. Prove that ϕ is minimal if and only if φ_1, φ_2 and φ_3 are holomorphic functions.

(iii) Consider the immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\phi(u,v) = (u - u^3/3 + uv^2,\ -v + v^3/3 - u^2 v,\ u^2 - v^2).$$

Find φ_1, φ_2 and φ_3. Show that ϕ is an isothermal parametrization of a minimal surface.

25J Probability and Measure

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel-measurable. State Fubini's theorem for the double integral

$$\int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} f(x,y)\ dx\, dy\ .$$

Let $0 < a < b$. Show that the function

$$f(x,y) = \begin{cases} e^{-xy} & \text{if } \ x \in (0,\infty),\ \ y \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

is measurable and integrable on $\mathbb{R}^2$.

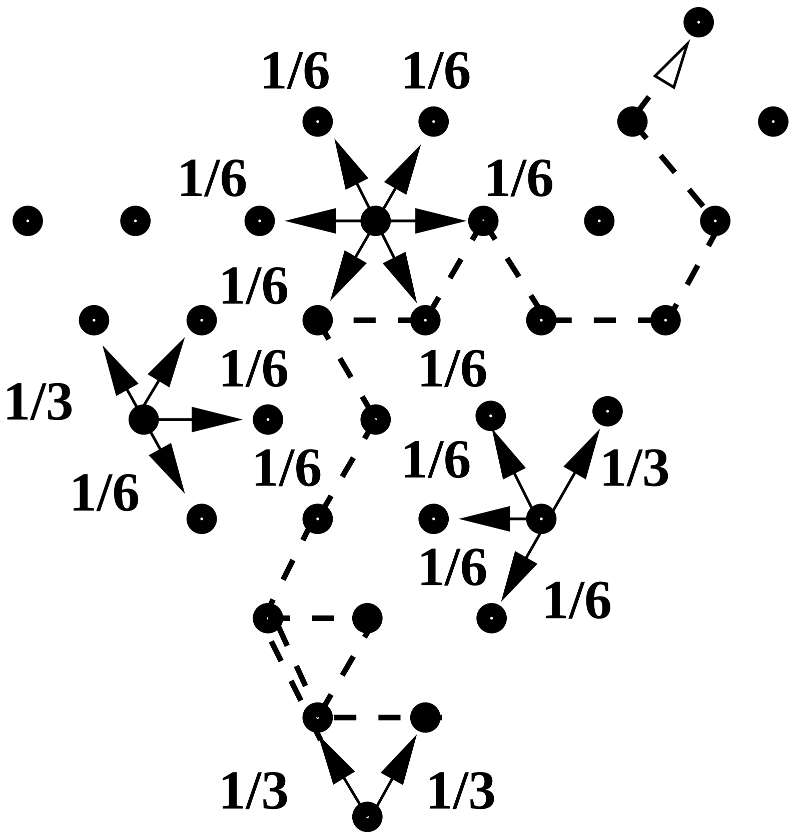
Evaluate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\ dx$$

by Fubini's theorem or otherwise.

26I Applied Probability

A particle performs a continuous-time nearest neighbour random walk on a regular triangular lattice inside an angle $\pi/3$, starting from the corner. See the diagram below. The jump rates are $1/3$ from the corner and $1/6$ in each of the six directions if the particle is inside the angle. However, if the particle is on the edge of the angle, the rate is $1/3$ along the edge away from the corner and $1/6$ to each of three other neighbouring sites in the angle. See the diagram below, where a typical trajectory is also shown.



The particle position at time $t \geqslant 0$ is determined by its vertical level V_t and its horizontal position G_t. For $k \geqslant 0$, if $V_t = k$ then $G_t = 0, \ldots, k$. Here $1, \ldots, k-1$ are positions inside, and 0 and k positions on the edge of the angle, at vertical level k.

Let J_1^V, J_2^V, ... be the times of subsequent jumps of process (V_t) and consider the embedded discrete-time Markov chains

$$Y_n^{\text{in}} = \left(\widehat{G}_n^{\text{in}}, \widehat{V}_n\right) \text{ and } Y_n^{\text{out}} = \left(\widehat{G}_n^{\text{out}}, \widehat{V}_n\right)$$

where $\widehat{V}_n$ is the vertical level immediately after time J_n^V, $\widehat{G}_n^{\text{in}}$ is the horizontal position immediately after time J_n^V, and $\widehat{G}_n^{\text{out}}$ is the horizontal position immediately before time J_{n+1}^V.

(a) Assume that $(\widehat{V}_n)$ is a Markov chain with transition probabilities

$$\mathbb{P}\big(\widehat{V}_n = k+1 \big| \widehat{V}_{n-1} = k\big) = \frac{k+2}{2(k+1)}, \ \ \mathbb{P}\big(\widehat{V}_n = k-1 \big| \widehat{V}_{n-1} = k\big) = \frac{k}{2(k+1)},$$

and that (V_t) is a continuous-time Markov chain with rates

$$q_{kk-1} = \frac{k}{3(k+1)}, \ \ q_{kk} = -\frac{2}{3}, \ \ q_{kk+1} = \frac{k+2}{3(k+1)}.$$

[*You will be asked to justify these assumptions in part (b) of the question.*] Determine whether the chains $(\widehat{V}_n)$ and (V_t) are transient, positive recurrent or null recurrent.

(b) Now assume that, conditional on $\widehat{V}_n = k$ and previously passed vertical levels, the horizontal positions $\widehat{G}_n^{\text{in}}$ and $\widehat{G}_n^{\text{out}}$ are uniformly distributed on $\{0, \dots, k\}$. In other words, for all attainable values k, k_{n-1}, …, k_1 and for all $i = 0, \dots, k$,

$$\begin{aligned} &\mathbb{P}\big(\widehat{G}_n^{\text{in}} = i \big| \widehat{V}_n = k, \widehat{V}_{n-1} = k_{n-1}, \dots, \widehat{V}_1 = k_1, \widehat{V}_0 = 0\big) \\ &\quad = \mathbb{P}\big(\widehat{G}_n^{\text{out}} = i \big| \widehat{V}_n = k, \widehat{V}_{n-1} = k_{n-1}, \dots, \widehat{V}_1 = k_1, \widehat{V}_0 = 0\big) = \frac{1}{k+1}\,. \end{aligned} \qquad (*)$$

Deduce that $(\widehat{V}_n)$ and (V_t) are indeed Markov chains with transition probabilities and rates as in (a).

(c) Finally, prove property $(*)$.

27I Principles of Statistics

A group of n hospitals is to be 'appraised'; the 'performance' θ_i of hospital i has a $N(0, 1/\tau)$ prior distribution, different hospitals being independent. The 'performance' cannot be measured directly, so an expensive firm of management consultants has been hired to arrive at each hospital's Standardised Index of Quality [SIQ], this being a number X_i for hospital i related to θ_i by the commercially-sensitive formula

$$X_i = \theta_i + \varepsilon_i,$$

where the ε_i are independent with common $N(0, 1/\tau_\varepsilon)$ distribution.

(i) Assume that τ and τ_ε are known. What is the posterior distribution of θ given X? Suppose that hospital j was the hospital with the lowest SIQ, with a value $X_j = x$; conditional on X, what is the distribution of θ_j?

(ii) Now, instead of assuming τ and τ_ε known, suppose that τ has a Gamma prior with parameters (α, β), density

$$f(t) = (\beta t)^{\alpha-1} \beta e^{-\beta t} / \Gamma(\alpha)$$

for known α and β, and that $\tau_\varepsilon = \kappa\tau$, where κ is a known constant. Find the posterior distribution of (θ, τ) given X. Comment briefly on the form of the distribution.

28J Stochastic Financial Models

(a) In the context of the Black–Scholes formula, let S_0 be spot price, K be strike price, T be time to maturity, and assume constant interest rate r, volatility σ and absence of dividends. Write down explicitly the prices of a European call and put,

$$EC\left(S_0, K, \sigma, r, T\right) \text{ and } EP\left(S_0, K, \sigma, r, T\right).$$

Use Φ for the normal distribution function. [*No proof is required.*]

(b) From here on assume $r = 0$. Keeping T, σ fixed, we shorten the notation to $EC\left(S_0, K\right)$ and similarly for EP. Show that *put-call symmetry* holds:

$$EC\left(S_0, K\right) = EP\left(K, S_0\right).$$

Check homogeneity: for every real $\alpha > 0$

$$EC\left(\alpha S_0, \alpha K\right) = \alpha EC\left(S_0, K\right).$$

(c) Show that the price of a down-and-out European call with strike $K < S_0$ and barrier $B \leqslant K$ is given by

$$EC\left(S_0, K\right) - \frac{S_0}{B} EC\left(\frac{B^2}{S_0}, K\right).$$

(d)

(i) Specialize the last expression to $B = K$ and simplify.

(ii) Answer a popular interview question in investment banks: What is the fair value of a down-and-out call given that $S_0 = 100$, $B = K = 80$, $\sigma = 20\%$, $r = 0$, $T = 1$? Identify the corresponding hedge. [*It may be helpful to compute Delta first.*]

(iii) Does this hedge work beyond the Black–Scholes model? When does it fail?

29I Optimization and Control

A continuous-time control problem is defined in terms of state variable $x(t) \in \mathbb{R}^n$ and control $u(t) \in \mathbb{R}^m$, $0 \leqslant t \leqslant T$. We desire to minimize $\int_0^T c(x,t)\,dt + K(x(T))$, where T is fixed and $x(T)$ is unconstrained. Given $x(0)$ and $\dot{x} = a(x,u)$, describe further boundary conditions that can be used in conjunction with Pontryagin's maximum principle to find x, u and the adjoint variables $\lambda_1, \ldots, \lambda_n$.

Company 1 wishes to steal customers from Company 2 and maximize the profit it obtains over an interval $[0,T]$. Denoting by $x_i(t)$ the number of customers of Company i, and by $u(t)$ the advertising effort of Company 1, this leads to a problem

$$\text{minimize} \int_0^T \big[x_2(t) + 3u(t)\big]\,dt\,,$$

where $\dot{x}_1 = ux_2$, $\dot{x}_2 = -ux_2$, and $u(t)$ is constrained to the interval $[0,1]$. Assuming $x_2(0) > 3/T$, use Pontryagin's maximum principle to show that the optimal advertising policy is bang-bang, and that there is just one change in advertising effort, at a time t^*, where

$$3\,e^{t^*} = x_2(0)(T - t^*)\,.$$

30C Partial Differential Equations

Write down the solution of the three-dimensional wave equation

$$u_{tt} - \Delta u = 0\,, \qquad u(0,x) = 0\,, \quad u_t(0,x) = g(x)\,,$$

for a Schwartz function g. Here Δ is taken in the variables $x \in \mathbb{R}^3$ and $u_t = \partial u/\partial t$ etc. State the "strong" form of Huygens principle for this solution. Using the method of descent, obtain the solution of the corresponding problem in two dimensions. State the "weak" form of Huygens principle for this solution.

Let $u \in C^2([0,T] \times \mathbb{R}^3)$ be a solution of

$$u_{tt} - \Delta u + |x|^2 u = 0\,, \qquad u(0,x) = 0\,, \quad u_t(0,x) = 0\,. \tag{$*$}$$

Show that

$$\partial_t e + \boldsymbol{\nabla} \cdot \mathbf{p} = 0\,, \tag{$**$}$$

where

$$e = \tfrac{1}{2}\big({u_t}^2 + |\boldsymbol{\nabla} u|^2 + |x|^2 u^2\big), \qquad \text{and} \qquad \mathbf{p} = -u_t \boldsymbol{\nabla} u\,.$$

Hence deduce, by integration of $(**)$ over the region

$$K = \big\{(t,x) : 0 \leqslant t \leqslant t_0 - a \leqslant t_0,\ \ |x - x_0| \leqslant t_0 - t\big\}$$

or otherwise, that $(*)$ satisfies the weak Huygens principle.

31A Asymptotic Methods

Consider the differential equation

$$\frac{d^2 w}{dx^2} = q(x) w\,,$$

where $q(x) \geqslant 0$ in an interval (a, ∞). Given a solution $w(x)$ and a further smooth function $\xi(x)$, define

$$W(x) = [\xi'(x)]^{1/2} w(x)\,.$$

Show that, when ξ is regarded as the independent variable, the function $W(\xi)$ obeys the differential equation

$$\frac{d^2 W}{d\xi^2} = \left\{ \dot{x}^2 q(x) + \dot{x}^{1/2} \frac{d^2}{d\xi^2} [\dot{x}^{-1/2}] \right\} W, \qquad (*)$$

where $\dot{x}$ denotes $dx/d\xi$.

Taking the choice

$$\xi(x) = \int q^{1/2}(x) dx\,,$$

show that equation $(*)$ becomes

$$\frac{d^2 W}{d\xi^2} = (1 + \phi) W\,,$$

where

$$\phi = -\frac{1}{q^{3/4}} \frac{d^2}{dx^2} \left(\frac{1}{q^{1/4}} \right).$$

In the case that ϕ is negligible, deduce the Liouville–Green approximate solutions

$$w_{\pm} = q^{-1/4} \exp\Big(\pm \int q^{1/2} dx \Big)\,.$$

Consider the Whittaker equation

$$\frac{d^2 w}{dx^2} = \left[\frac{1}{4} + \frac{s(s-1)}{x^2} \right] w\,,$$

where s is a real constant. Show that the Liouville–Green approximation suggests the existence of solutions $w_{A,B}(x)$ with asymptotic behaviour of the form

$$w_A \sim \exp(x/2) \left(1 + \sum_{n=1}^{\infty} a_n x^{-n} \right), \qquad w_B \sim \exp(-x/2) \left(1 + \sum_{n=1}^{\infty} b_n x^{-n} \right)$$

as $x \to \infty$.

Given that these asymptotic series may be differentiated term-by-term, show that

$$a_n = \frac{(-1)^n}{n!} (s-n)(s-n+1) \ldots (s+n-1)\,.$$

32D Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + \lambda V(t)\,,$$

where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|a\rangle$ and $|b\rangle$ be eigenstates of H_0 with distinct eigenvalues E_a and E_b respectively. Show that if the system is initially in state $|a\rangle$ then the probability of measuring it to be in state $|b\rangle$ after a time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_0^t dt'\langle b|V(t')|a\rangle e^{i(E_b-E_a)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\;.$$

Deduce that if $V(t) = e^{-\mu t/\hbar}W$, where W is a time-independent operator and μ is a positive constant, then the probability for such a transition to have occurred after a very long time is approximately

$$\frac{\lambda^2}{\mu^2+(E_b-E_a)^2}|\langle b|W|a\rangle|^2\;.$$

33B Applications of Quantum Mechanics

A semiconductor has a valence energy band with energies $E \leqslant 0$ and density of states $g_v(E)$, and a conduction energy band with energies $E \geqslant E_g$ and density of states $g_c(E)$. Assume that $g_v(E) \sim A_v(-E)^{\frac{1}{2}}$ as $E \to 0$, and that $g_c(E) \sim A_c(E - E_g)^{\frac{1}{2}}$ as $E \to E_g$. At zero temperature all states in the valence band are occupied and the conduction band is empty. Let p be the number of holes in the valence band and n the number of electrons in the conduction band at temperature T. Under suitable approximations derive the result

$$pn = N_v N_c e^{-E_g/kT}$$

where

$$N_v = \tfrac{1}{2}\sqrt{\pi}A_v(kT)^{\frac{3}{2}}\,, \qquad N_c = \tfrac{1}{2}\sqrt{\pi}A_c(kT)^{\frac{3}{2}}\,.$$

Briefly describe how a semiconductor may conduct electricity but with a conductivity that is strongly temperature dependent.

Describe how doping of the semiconductor leads to $p \neq n$. A pn junction is formed between an n-type semiconductor, with N_d donor atoms, and a p-type semiconductor, with N_a acceptor atoms. Show that there is a potential difference $V_{np} = \Delta E/|e|$ across the junction, where e is the electron charge, and

$$\Delta E = E_g - kT \ln \frac{N_v N_c}{N_d N_a}\,.$$

Two semiconductors, one p-type and one n-type, are joined to make a closed circuit with two pn junctions. Explain why a current will flow around the circuit if the junctions are at different temperatures.

[*The Fermi–Dirac distribution function at temperature T and chemical potential μ is $\dfrac{g(E)}{e^{(E-\mu)/kT}+1}$, where $g(E)$ is the number of states with energy E.*

Note that $\displaystyle\int_0^\infty x^{\frac{1}{2}} e^{-x}\, dx = \tfrac{1}{2}\sqrt{\pi}$.]

34D Statistical Physics

Write down an expression for the partition function of a classical particle of mass m moving in three dimensions in a potential $U(\mathbf{x})$ and in equilibrium with a heat bath at temperature T.

A system of N non-interacting classical particles is placed in the potential

$$U(\mathbf{x}) \;=\; \frac{(x^2+y^2+z^2)^n}{V^{2n/3}}\,,$$

where n is a positive integer. The gas is in equilibrium at temperature T. Using a suitable rescaling of variables, show that the free energy F is given by

$$\frac{F}{N} \;=\; -kT\left(\log V + \frac{3}{2}\frac{n+1}{n}\log kT + \log I_n\right)\,,$$

where

$$I_n = \left(\frac{2m\pi}{h^2}\right)^{3/2}\int_0^\infty 4\pi u^2 e^{-u^{2n}}\,du\,.$$

Regarding V as an external parameter, find the thermodynamic force P, conjugate to V, exerted by this system. Find the equation of state and compare with that of an ideal gas confined in a volume V.

Derive expressions for the entropy S, the internal energy E and the total heat capacity C_V at constant V.

Show that for all n the total heat capacity at constant P is given by

$$C_P \;=\; C_V + Nk\,.$$

[*Note that* $\displaystyle\int_0^\infty u^2 e^{-u^2/2}\,du \;=\; \sqrt{\frac{\pi}{2}}$.]

35B Electrodynamics

In Ginzburg–Landau theory, superconductivity is due to "supercarriers" of mass m_s and charge q_s, which are described by a macroscopic wavefunction ψ with "Mexican hat" potential

$$V = \alpha(T)|\psi|^2 + \frac{1}{2}\beta|\psi|^4 .$$

Here, $\beta > 0$ is constant and $\alpha(T)$ is a function of temperature T such that $\alpha(T) > 0$ for $T > T_c$ but $\alpha(T) < 0$ for $T < T_c$, where T_c is a critical temperature. In the presence of a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, the total energy of the superconducting system is

$$E[\psi, \psi^*, \mathbf{A}] = \int d^3x \left[\frac{1}{2\mu_0} A_{k,j}\big(A_{k,j} - A_{j,k}\big) + \frac{\hbar^2}{2m_s}\left|\psi_{,k} + i\frac{q_s}{\hbar}A_k\psi\right|^2 + V \right] .$$

Use this to derive the equations

$$-\frac{\hbar^2}{2m_s}\big(\boldsymbol{\nabla} - i\frac{q_s}{\hbar}\mathbf{A}\big)^2\psi + \big(\alpha + \beta|\psi|^2\big)\psi = 0 \qquad (*)$$

and

$$\boldsymbol{\nabla} \times \mathbf{B} \equiv \boldsymbol{\nabla}\big(\boldsymbol{\nabla} \cdot \mathbf{A}\big) - \boldsymbol{\nabla}^2\mathbf{A} = \mu_0\,\mathbf{j} \,, \qquad (\dagger)$$

where

$$\begin{aligned} \mathbf{j} &= -\frac{iq_s\hbar}{2m_s}\big(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\big) - \frac{q_s^2}{m_s}|\psi|^2\mathbf{A} \\ &= \frac{q_s}{2m_s}\left[\psi^*\left(-i\hbar\boldsymbol{\nabla} - q_s\mathbf{A}\right)\psi + \psi\left(i\hbar\boldsymbol{\nabla} - q_s\mathbf{A}\right)\psi^*\right] . \end{aligned}$$

Suppose that we write the wavefunction as

$$\psi = \sqrt{n_s}\, e^{i\theta} \,,$$

where n_s is the (real positive) supercarrier density and θ is a real phase function. Given that

$$\left(\boldsymbol{\nabla} - \frac{iq_s}{\hbar}\,\mathbf{A}\right)\psi = 0 \,,$$

show that n_s is constant and that $\hbar\boldsymbol{\nabla}\theta = q_s\mathbf{A}$. Given also that $T < T_c$, deduce that $(*)$ allows such solutions for a certain choice of n_s, which should be determined. Verify that your results imply $\mathbf{j} = \mathbf{0}$. Show also that $\mathbf{B} = \mathbf{0}$ and hence that $(\dagger)$ is solved.

Let $\mathcal{S}$ be a surface within the superconductor with closed boundary $\mathcal{C}$. Show that the magnetic flux through $\mathcal{S}$ is

$$\Phi \equiv \int_{\mathcal{S}} \mathbf{B} \cdot \mathbf{dS} = \frac{\hbar}{q_s}\big[\theta\big]_{\mathcal{C}} \,.$$

Discuss, briefly, flux quantization.

36C General Relativity

State clearly, but do not prove, Birkhoff's Theorem about spherically symmetric spacetimes. Let (r, θ, ϕ) be standard spherical polar coordinates and define $F(r) = 1 - 2M/r$, where M is a constant. Consider the metric

$$ds^2 = \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - F(r)\, dt^2.$$

Explain carefully why this is appropriate for the region outside a spherically symmetric star that is collapsing to form a black hole.

By considering radially infalling timelike geodesics $x^a = (r(\tau), 0, 0, t(\tau))$, where τ is proper time along the curve, show that a freely falling observer will reach the event horizon after a finite proper time. Show also that a distant observer would see the horizon crossing only after an infinite time.

37E Fluid Dynamics II

Consider flow of an incompressible fluid of uniform density ρ and dynamic viscosity μ. Show that the rate of viscous dissipation per unit volume is given by

$$\Phi = 2\mu e_{ij}e_{ij}\,,$$

where e_{ij} is the strain rate.

Determine expressions for e_{ij} and Φ when the flow is irrotational with velocity potential ϕ. Hence determine the rate of viscous dissipation, averaged over a wave period $2\pi/\omega$, for an irrotational two-dimensional surface wave of wavenumber k and small amplitude $a \ll k^{-1}$ in a fluid of very small viscosity $\mu \ll \rho\omega/k^2$ and great depth $H \gg 1/k$.

[*You may use without derivation that in deep water a linearised wave with surface displacement* $\eta = a\cos\,(kx - wt)$ *has velocity potential* $\phi = -(\omega a/k)e^{-kz}\sin\,(kx - \omega t)$.]

Calculate the depth-integrated time-averaged kinetic energy per wavelength. Assuming that the average potential energy is equal to the average kinetic energy, show that the total wave energy decreases to leading order like $e^{-\gamma t}$, where

$$\gamma = 4\mu k^2/\rho\ .$$

38E Waves

Starting from the equations of conservation of mass and momentum for an inviscid compressible fluid, show that for small perturbations about a state of rest and uniform density the velocity is irrotational and the velocity potential satisfies the wave equation. Identify the sound speed c_0.

Define the acoustic energy density and acoustic energy flux, and derive the equation for conservation of acoustic energy.

Show that in any (not necessarily harmonic) acoustic plane wave of wavenumber $\mathbf{k}$ the kinetic and potential energy densities are equal and that the acoustic energy is transported with velocity $c_0\widehat{\mathbf{k}}$.

Calculate the kinetic and potential energy densities for a spherically symmetric outgoing wave. Are they equal?

39A Numerical Analysis

An $n \times n$ *skew-symmetric* matrix A is converted into an upper-Hessenberg form B, say, by Householder reflections.

(a) Describe each step of the procedure and observe that B is tridiagonal. Your algorithm should take advantage of the special form of A to reduce the number of computations.

(b) Compare the cost (counting only products and looking only at the leading term) of converting a skew-symmetric and a symmetric matrix to an upper-Hessenberg form using Householder reflections.

END OF PAPER