UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 8 June 2005 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Linear Algebra

Let Ω be the set of all 2×2 matrices of the form $\alpha = aI + bJ + cK + dL$, where a, b, c, d are in $\mathbf{R}$, and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ L = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (i^2 = -1).$$

Prove that Ω is closed under multiplication and determine its dimension as a vector space over $\mathbf{R}$. Prove that

$$(aI + bJ + cK + dL)\,(aI - bJ - cK - dL) = (a^2 + b^2 + c^2 + d^2)I\,,$$

and deduce that each non-zero element of Ω is invertible.

2C Groups, Rings and Modules

Define an automorphism of a group G, and the natural group law on the set $\mathrm{Aut}(G)$ of all automorphisms of G. For each fixed h in G, put $\psi(h)(g) = hgh^{-1}$ for all g in G. Prove that $\psi(h)$ is an automorphism of G, and that ψ defines a homomorphism from G into $\mathrm{Aut}(G)$.

3B Analysis II

Define uniform continuity for a real-valued function defined on an interval in $\mathbf{R}$.

Is a uniformly continuous function on the interval $(0, 1)$ necessarily bounded?

Is $1/x$ uniformly continuous on $(0, 1)$?

Is $\sin(1/x)$ uniformly continuous on $(0, 1)$?

Justify your answers.

4A Metric and Topological Spaces

Let X be a topological space. Suppose that $U_1, U_2, \ldots$ are connected subsets of X with $U_j \cap U_1$ non-empty for all $j > 0$. Prove that

$$W = \bigcup_{j>0} U_j$$

is connected. If each U_j is path-connected, prove that W is path-connected.

5E Methods

Consider the differential equation for $x(t)$ in $t > 0$

$$\ddot{x} - k^2 x = f(t),$$

subject to boundary conditions $x(0) = 0$, and $\dot{x}(0) = 0$. Find the Green function $G(t, t')$ such that the solution for $x(t)$ is given by

$$x(t) = \int_0^t G(t, t') f(t')\, dt'.$$

6H Electromagnetism

Write down Maxwell's equations in the presence of a charge density ρ and current density $\mathbf{J}$. Show that it is necessary that $\rho, \mathbf{J}$ satisfy a conservation equation.

If $\rho, \mathbf{J}$ are zero outside a fixed region V show that the total charge inside V is a constant and also that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathbf{x}\rho \ \mathrm{d}^3x = \int_V \mathbf{J} \ \mathrm{d}^3x\,.$$

7G Special Relativity

Bob and Alice are twins. Bob accelerates rapidly away from Earth in a rocket that travels in a straight line until it reaches a velocity v relative to the Earth. It then travels with constant v for a long time before reversing its engines and decelerating rapidly until it is travelling at a velocity $-v$ relative to the Earth. After a further long period of time the rocket returns to Earth, decelerating rapidly until it is at rest. Alice remains on Earth throughout. Sketch the space-time diagram that describes Bob's world-line in Alice's rest frame, assuming that the periods of acceleration and deceleration are negligibly small compared to the total time, explain carefully why Bob ages less than Alice between his departure and his return and show that

$$\Delta t_B = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t_A\,,$$

where Δt_B is the time by which Bob has aged and Δt_A is the time by which Alice has aged.

Indicate on your diagram how Bob sees Alice aging during his voyage.

8E Fluid Dynamics

For a steady flow of an incompressible fluid of density ρ, show that

$$\mathbf{u} \times \boldsymbol{\omega} = \nabla H \,,$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity and H is to be found. Deduce that H is constant along streamlines.

Now consider a flow in the xy-plane described by a streamfunction $\psi(x, y)$. Evaluate $\mathbf{u} \times \boldsymbol{\omega}$ and deduce from $H = H(\psi)$ that

$$\frac{dH}{d\psi} + \omega = 0.$$

9D Optimization

Explain what is meant by a two-person zero-sum game with payoff matrix $A = (a_{ij})$.

Show that the problems of the two players may be expressed as a dual pair of linear programming problems. State without proof a set of sufficient conditions for a pair of strategies for the two players to be optimal.

SECTION II

10C Linear Algebra

(i) Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in $\mathbf{C}$. Define the determinant of A, the cofactor of each a_{ij}, and the adjugate matrix $\mathrm{adj}(A)$. Assuming the expansion of the determinant of a matrix in terms of its cofactors, prove that

$$\mathrm{adj}(A)\, A = \det(A) I_n \,,$$

where I_n is the $n \times n$ identity matrix.

(ii) Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Show the eigenvalues of A are $\pm 1, \pm i$, where $i^2 = -1$, and determine the diagonal matrix to which A is similar. For each eigenvalue, determine a non-zero eigenvector.

11C Groups, Rings and Modules

Let A be the abelian group generated by two elements x, y, subject to the relation $6x + 9y = 0$. Give a rigorous explanation of this statement by defining A as an appropriate quotient of a free abelian group of rank 2. Prove that A itself is not a free abelian group, and determine the exact structure of A.

12A Geometry

Let U be an open subset of $\mathbf{R}^2$ equipped with a Riemannian metric. For $\gamma : [0,1] \to U$ a smooth curve, define what is meant by its *length* and *energy.* Prove that *length*$(\gamma)^2 \leq$ *energy*(γ), with equality if and only if $\dot\gamma$ has constant norm with respect to the metric.

Suppose now U is the upper half plane model of the hyperbolic plane, and P, Q are points on the positive imaginary axis. Show that a smooth curve γ joining P and Q represents an absolute minimum of the length of such curves if and only if $\gamma(t) = i\, v(t)$, with v a smooth monotonic real function.

Suppose that a smooth curve γ joining the above points P and Q represents a stationary point for the energy under proper variations; deduce from an appropriate form of the Euler–Lagrange equations that γ must be of the above form, with $\dot v / v$ constant.

13B Analysis II

Use the standard metric on $\mathbf{R}^n$ in this question.

(i) Let A be a nonempty closed subset of $\mathbf{R}^n$ and y a point in $\mathbf{R}^n$. Show that there is a point $x \in A$ which minimizes the distance to y, in the sense that $d(x,y) \leqslant d(a,y)$ for all $a \in A$.

(ii) Suppose that the set A in part (i) is convex, meaning that A contains the line segment between any two of its points. Show that point $x \in A$ described in part (i) is unique.

14F Complex Analysis or Complex Methods

Let $F = P/Q$ be a rational function, where $\deg Q \geqslant \deg P + 2$ and Q has no real zeros. Using the calculus of residues, write a general expression for

$$\int_{-\infty}^{\infty} F(x) e^{ix} dx$$

in terms of residues and briefly sketch its proof.

Evaluate explicitly the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{4 + x^4} dx \, .$$

15E Methods

Write down the Euler–Lagrange equation for the variational problem for $r(z)$

$$\delta \int_{-h}^{h} F(z, r, r')\, dz = 0,$$

with boundary conditions $r(-h) = r(h) = R$, where R is a given positive constant. Show that if F does not depend explicitly on z, i.e. $F = F(r, r')$, then the equation has a first integral

$$F - r' \frac{\partial F}{\partial r'} = \frac{1}{k},$$

where k is a constant.

An axisymmetric soap film $r(z)$ is formed between two circular rings $r = R$ at $z = \pm H$. Find the equation governing the shape which minimizes the surface area. Show that the shape takes the form

$$r(z) = k^{-1} \cosh kz.$$

Show that there exist no solution if $R/H < \sinh A$, where A is the unique positive solution of $A = \coth A$.

16G Quantum Mechanics

A particle of mass m moving in a one-dimensional harmonic oscillator potential satisfies the Schrödinger equation

$$H\,\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\,\Psi(x,t)\,,$$

where the Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\,\omega^2\,x^2\,.$$

The operators a and $a^\dagger$ are defined by

$$a = \frac{1}{\sqrt{2}}\left(\beta x + \frac{i}{\beta\hbar}p\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\beta x - \frac{i}{\beta\hbar}p\right),$$

where $\beta = \sqrt{m\omega/\hbar}$ and $p = -i\hbar\partial/\partial x$ is the usual momentum operator. Show that $[a, a^\dagger] = 1$.

Express x and p in terms of a and $a^\dagger$ and, hence or otherwise, show that H can be written in the form

$$H = \left(a^\dagger a + \tfrac{1}{2}\right)\,\hbar\omega\,.$$

Show, for an arbitrary wave function Ψ, that $\int dx\,\Psi^*\,H\,\Psi \geq \frac{1}{2}\hbar\omega$ and hence that the energy of any state satisfies the bound

$$E \geq \tfrac{1}{2}\hbar\omega\,.$$

Hence, or otherwise, show that the ground state wave function satisfies $a\Psi_0 = 0$ and that its energy is given by

$$E_0 = \tfrac{1}{2}\hbar\omega\,.$$

By considering H acting on $a^\dagger\,\Psi_0$, $(a^\dagger)^2\,\Psi_0$, and so on, show that states of the form

$$(a^\dagger)^n\,\Psi_0$$

$(n > 0)$ are also eigenstates and that their energies are given by $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$.

17H Electromagnetism

Assume the magnetic field

$$\mathbf{B}(\mathbf{x}) = b(\mathbf{x} - 3\,\hat{\mathbf{z}}\;\hat{\mathbf{z}}\cdot\mathbf{x})\,, \qquad (*)$$

where $\hat{\mathbf{z}}$ is a unit vector in the vertical direction. Show that this satisfies the expected equations for a static magnetic field in vacuum.

A circular wire loop, of radius a, mass m and resistance R, lies in a horizontal plane with its centre on the z-axis at a height z and there is a magnetic field given by $(*)$. Calculate the magnetic flux arising from this magnetic field through the loop and also the force acting on the loop when a current I is flowing around the loop in a clockwise direction about the z-axis.

Obtain an equation of motion for the height $z(t)$ when the wire loop is falling under gravity. Show that there is a solution in which the loop falls with constant speed v which should be determined. Verify that in this situation the rate at which heat is generated by the current flowing in the loop is equal to the rate of loss of gravitational potential energy. What happens when $R \to 0$?

18F Numerical Analysis

(a) Let $\{Q_n\}_{n\geqslant 0}$ be a set of polynomials orthogonal with respect to some inner product $(\,\cdot\,,\,\cdot\,)$ in the interval $[a,b]$. Write explicitly the least-squares approximation to $f \in C[a,b]$ by an nth-degree polynomial in terms of the polynomials $\{Q_n\}_{n\geqslant 0}$.

(b) Let an inner product be defined by the formula

$$(g,h) = \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} g(x)h(x)dx.$$

Determine the nth degree polynomial approximation of $f(x) = (1-x^2)^{\frac{1}{2}}$ with respect to this inner product as a linear combination of the underlying orthogonal polynomials.

19D Statistics

Let $X_1, \ldots, X_n$ be a random sample from a probability density function $f(x \mid \theta)$, where θ is an unknown real-valued parameter which is assumed to have a prior density $\pi(\theta)$. Determine the optimal Bayes point estimate $a(X_1, \ldots, X_n)$ of θ, in terms of the posterior distribution of θ given $X_1, \ldots, X_n$, when the loss function is

$$L(\theta, a) = \begin{cases} \gamma(\theta - a) & \text{when } \theta \geqslant a, \\ \delta(a - \theta) & \text{when } \theta \leqslant a, \end{cases}$$

where γ and δ are given positive constants.

Calculate the estimate explicitly in the case when $f(x \mid \theta)$ is the density of the uniform distribution on $(0, \theta)$ and $\pi(\theta) = e^{-\theta}\theta^n/n!$, $\theta > 0$.

20D Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ with state space $\{0, 1, 2, \ldots\}$ and transition probabilities given by

$$P_{i,j} = pq^{i-j+1}, \quad 0 < j \leqslant i+1, \quad \text{and} \quad P_{i,0} = q^{i+1} \quad \text{for} \quad i \geqslant 0,$$

with $P_{i,j} = 0$, otherwise, where $0 < p < 1$ and $q = 1 - p$.

For each $i \geqslant 1$, let

$$h_i = \mathbb{P}\left(X_n = 0, \text{for some } n \geqslant 0 \mid X_0 = i\right),$$

that is, the probability that the chain ever hits the state 0 given that it starts in state i. Write down the equations satisfied by the probabilities $\{h_i,\, i \geqslant 1\}$ and hence, or otherwise, show that they satisfy a second-order recurrence relation with constant coefficients. Calculate h_i for each $i \geqslant 1$.

Determine for each value of p, $0 < p < 1$, whether the chain is transient, null recurrent or positive recurrent and in the last case calculate the stationary distribution.

[*Hint: When the chain is positive recurrent, the stationary distribution is geometric.*]

END OF PAPER