UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 3 June 2005 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* **three** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***B*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

Solve the equation

$$\frac{dy}{dx} + 3x^2 y = x^2,$$

with $y(0) = a$, by use of an integrating factor or otherwise. Find $\lim_{x \to +\infty} y(x)$.

2B Differential Equations

Obtain the general solution of

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \qquad (*)$$

by using the indicial equation.

Introduce $z = \log x$ as a new independent variable and find an equivalent second order differential equation with constant coefficients. Determine the general solution of this new equation, and show that it is equivalent to the general solution of $(*)$ found previously.

3F Probability

Suppose $c \geqslant 1$ and X_c is a positive real-valued random variable with probability density

$$f_c(t) = A_c t^{c-1} e^{-t^c},$$

for $t > 0$, where A_c is a constant.

Find the constant A_c and show that, if $c > 1$ and $s, t > 0$,

$$\mathbb{P}[X_c \geqslant s + t \mid X_c \geqslant t] < \mathbb{P}[X_c \geqslant s].$$

[*You may assume the inequality* $(1+x)^c > 1 + x^c$ *for all* $x > 0$, $c > 1$.]

4F Probability

Describe the Poisson distribution characterised by parameter $\lambda > 0$. Calculate the mean and variance of this distribution in terms of λ.

Show that the sum of n independent random variables, each having the Poisson distribution with $\lambda = 1$, has a Poisson distribution with $\lambda = n$.

Use the central limit theorem to prove that

$$e^{-n}\left(1 + \frac{n}{1!} + \frac{n^2}{2!} + \ldots + \frac{n^n}{n!}\right) \to 1/2 \quad \text{as} \quad n \to \infty.$$

SECTION II

5B Differential Equations

Find two linearly independent solutions of the difference equation

$$X_{n+2} - 2\cos\theta X_{n+1} + X_n = 0\,,$$

for all values of $\theta \in (0,\pi)$. What happens when $\theta = 0$? Find two linearly independent solutions in this case.

Find $X_n(\theta)$ which satisfy the initial conditions

$$X_1 = 1, \qquad X_2 = 2,$$

for $\theta = 0$ and for $\theta \in (0,\pi)$. For every n, show that $X_n(\theta) \to X_n(0)$ as $\theta \to 0$.

6B Differential Equations

Find all power series solutions of the form $W = \sum_{n=0}^{\infty} a_n x^n$ to the equation

$$-W'' + 2xW' = EW\,,$$

for E a real constant.

Impose the condition $W(0) = 0$ and determine those values of E for which your power series gives polynomial solutions (i.e., $a_n = 0$ for n sufficiently large). Give the values of E for which the corresponding polynomials have degree less than 6, and compute these polynomials.

Hence, or otherwise, find a polynomial solution of

$$-W'' + 2xW' = x - \frac{4}{3}x^3 + \frac{4}{15}x^5\,,$$

satisfying $W(0) = 0$.

7B Differential Equations

The Cartesian coordinates (x, y) of a point moving in $\mathbb{R}^2$ are governed by the system

$$\begin{aligned}\frac{dx}{dt} &= -y + x(1 - x^2 - y^2)\,,\\ \frac{dy}{dt} &= x + y(1 - x^2 - y^2)\,.\end{aligned}$$

Transform this system of equations to polar coordinates (r, θ) and hence find all periodic solutions (i.e., closed trajectories) which satisfy $r =$ constant.

Discuss the large time behaviour of an arbitrary solution starting at initial point $(x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0)$. Summarize the motion using a phase plane diagram, and comment on the nature of any critical points.

8B Differential Equations

Define the Wronskian $W[u_1, u_2]$ for two solutions u_1, u_2 of the equation

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = 0$$

and obtain a differential equation which exhibits its dependence on x. Explain the relevance of the Wronskian to the linear independence of u_1 and u_2.

Consider the equation

$$x^2\frac{d^2y}{dx^2} - 2y = 0 \qquad (*)$$

and determine the dependence on x of the Wronskian $W[y_1, y_2]$ of two solutions y_1 and y_2. Verify that $y_1(x) = x^2$ is a solution of $(*)$ and use the Wronskian to obtain a second linearly independent solution.

9F Probability

Given a real-valued random variable X, we define $\mathbb{E}[e^{iX}]$ by

$$\mathbb{E}\left[e^{iX}\right] \equiv \mathbb{E}\left[\cos X\right] + i\,\mathbb{E}\left[\sin X\right] .$$

Consider a second real-valued random variable Y, independent of X. Show that

$$\mathbb{E}\left[e^{i(X+Y)}\right] = \mathbb{E}\left[e^{iX}\right]\mathbb{E}\left[e^{iY}\right] .$$

You gamble in a fair casino that offers you unlimited credit despite your initial wealth of 0. At every game your wealth increases or decreases by £1 with equal probability 1/2. Let W_n denote your wealth after the n^{th} game. For a fixed real number u, compute $\phi(u)$ defined by

$$\phi\left(u\right) = \mathbb{E}\left[e^{iuW_n}\right] .$$

Verify that the result is real-valued.

Show that for n even,

$$\mathbb{P}\left[W_n = 0\right] = \gamma \int_0^{\pi/2} \left[\cos u\right]^n du \,,$$

for some constant γ, which you should determine. What is $\mathbb{P}\left[W_n = 0\right]$ for n odd?

10F Probability

Alice and Bill fight a paint-ball duel. Nobody has been hit so far and they are both left with one shot. Being exhausted, they need to take a breath before firing their last shot. This takes A seconds for Alice and B seconds for Bill. Assume these times are exponential random variables with means $1/\alpha$ and $1/\beta$, respectively.

Find the distribution of the (random) time that passes by before the next shot is fired. What is its standard deviation? What is the probability that Alice fires the next shot?

Assume Alice has probability 1/2 of hitting whenever she fires whereas Bill never misses his target. If the next shot is a hit, what is the probability that it was fired by Alice?

11F Probability

Let (S,T) be uniformly distributed on $[-1,1]^2$ and define $R=\sqrt{S^2+T^2}$. Show that, conditionally on

$$R \leqslant 1\,,$$

the vector (S,T) is uniformly distributed on the unit disc. Let (R,Θ) denote the point (S,T) in polar coordinates and find its probability density function $f\,(r,\theta)$ for $r \in [0,1]$, $\theta \in [0,2\pi)$. Deduce that R and Θ are independent.

Introduce the new random variables

$$X=\frac{S}{R}\sqrt{-2\log\left(R^2\right)},\;\; Y=\frac{T}{R}\sqrt{-2\log\left(R^2\right)}\,,$$

noting that under the above conditioning, (S,T) are uniformly distributed on the unit disc. The pair (X,Y) may be viewed as a (random) point in $\mathbb{R}^2$ with polar coordinates (Q,Ψ). Express Q as a function of R and deduce its density. Find the joint density of (Q,Ψ). Hence deduce that X and Y are independent normal random variables with zero mean and unit variance.

12F Probability

Let $a_1,a_2,...,a_n$ be a ranking of the yearly rainfalls in Cambridge over the next n years: assume $a_1,a_2,...,a_n$ is a random permutation of $1,2,...,n$. Year k is called a record year if $a_i > a_k$ for all $i<k$ (thus the first year is always a record year). Let $Y_i = 1$ if year i is a record year and 0 otherwise.

Find the distribution of Y_i and show that $Y_1,Y_2,...,Y_n$ are independent and calculate the mean and variance of the number of record years in the next n years.

Find the probability that the second record year occurs at year i. What is the expected number of years until the second record year occurs?

END OF PAPER