UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2004

List of Courses

Algebra and Geometry
Analysis
Differential Equations
Dynamics
Numbers and Sets
Probability
Vector Calculus

1/I/1B **Algebra and Geometry**

The linear map $H : \mathbb{R}^3 \to \mathbb{R}^3$ represents reflection in the plane through the origin with normal $\mathbf{n}$, where $|\mathbf{n}| = 1$, and $\mathbf{n} = (n_1, n_2, n_3)$ referred to the standard basis. The map is given by $\mathbf{x} \mapsto \mathbf{x}' = \mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is a (3×3) matrix.

Show that

$$M_{ij} = \delta_{ij} - 2n_i n_j \,.$$

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors such that $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ is an orthonormal set. Show that

$$\mathbf{M}\mathbf{n} = -\mathbf{n}, \quad \mathbf{M}\mathbf{u} = \mathbf{u}, \quad \mathbf{M}\mathbf{v} = \mathbf{v}\,,$$

and find the matrix $\mathbf{N}$ which gives the mapping relative to the basis $(\mathbf{u}, \mathbf{v}, \mathbf{n})$.

1/I/2C **Algebra and Geometry**

Show that

$$\sum_{i=1}^{n} a_i b_i \leqslant \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2}$$

for any real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. Using this inequality, show that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of unit length in $\mathbb{R}^n$ then $|\mathbf{a} \cdot \mathbf{b}| \leqslant 1$.

1/II/5B **Algebra and Geometry**

The vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfies the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,,$$

where $\mathbf{A}$ is a (3×3) matrix and $\mathbf{b}$ is a (3×1) column vector. State the conditions under which this equation has (a) a unique solution, (b) an infinity of solutions, (c) no solution for $\mathbf{x}$.

Find all possible solutions for the unknowns x, y and z which satisfy the following equations:

$$\begin{aligned} x + y + z &= 1 \\ x + y + \lambda z &= 2 \\ x + 2y + \lambda z &= 4\,, \end{aligned}$$

in the cases (a) $\lambda = 0$, and (b) $\lambda = 1$.

1/II/6A **Algebra and Geometry**

Express the product $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ in suffix notation and thence prove that the result is zero.

Silver Beard the space pirate believed people relied so much on space-age navigation techniques that he could safely write down the location of his treasure using the ancient art of vector algebra. Spikey the space jockey thought he could follow the instructions, by moving by the sequence of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ one stage at a time. The vectors (expressed in 1000 parsec units) were defined as follows:

1. Start at the centre of the galaxy, which has coordinates $(0, 0, 0)$.

2. Vector $\mathbf{a}$ has length $\sqrt{3}$, is normal to the plane $x + y + z = 1$ and is directed into the positive quadrant.

3. Vector $\mathbf{b}$ is given by $\mathbf{b} = (\mathbf{a} \cdot \mathbf{m})\mathbf{a} \times \mathbf{m}$, where $\mathbf{m} = (2, 0, 1)$.

4. Vector $\mathbf{c}$ has length $2\sqrt{2}$, is normal to $\mathbf{a}$ and $\mathbf{b}$, and moves you closer to the x axis.

5. Vector $\mathbf{d} = (1, -2, 2)$.

6. Vector $\mathbf{e}$ has length $\mathbf{a} \cdot \mathbf{b}$. Spikey was initially a little confused with this one, but then realised the orientation of the vector did not matter.

7. Vector $\mathbf{f}$ has unknown length but is parallel to $\mathbf{m}$ and takes you to the treasure located somewhere on the plane $2x - y + 4z = 10$.

Determine the location of the way-points Spikey will use and thence the location of the treasure.

1/II/7A **Algebra and Geometry**

Simplify the fraction

$$\zeta = \cfrac{1}{\bar{z} + \cfrac{1}{z + \cfrac{1}{\bar{z}}}},$$

where $\bar{z}$ is the complex conjugate of z. Determine the geometric form that satisfies

$$\mathrm{Re}(\zeta) = \mathrm{Re}\left(\frac{z + \frac{1}{4}}{|z|^2}\right).$$

Find solutions to

$$\mathrm{Im}(\log z) = \frac{\pi}{3}$$

and

$$z^2 = x^2 - y^2 + 2ix,$$

where $z = x + iy$ is a complex variable. Sketch these solutions in the complex plane and describe the region they enclose. Derive complex equations for the circumscribed and inscribed circles for the region. [The circumscribed circle is the circle that passes through the vertices of the region and the inscribed circle is the largest circle that fits within the region.]

1/II/8C **Algebra and Geometry**

(i) The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfy $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3 \neq 0$. Are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(ii) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$ have the property that every subset comprising $(n-1)$ of the vectors is linearly independent. Are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(iii) For each value of t in the range $0 \leqslant t < 1$, give a construction of a linearly independent set of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfying

$$\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij} + t(1 - \delta_{ij}),$$

where δ_{ij} is the Kronecker delta.

3/I/1D **Algebra and Geometry**

State Lagrange's Theorem.

Show that there are precisely two non-isomorphic groups of order 10. [You may assume that a group whose elements are all of order 1 or 2 has order 2^k.]

3/I/2D Algebra and Geometry

Define the Möbius group, and describe how it acts on $\mathbb{C} \cup \{\infty\}$.

Show that the subgroup of the Möbius group consisting of transformations which fix 0 and ∞ is isomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Now show that the subgroup of the Möbius group consisting of transformations which fix 0 and 1 is also isomorphic to $\mathbb{C}^*$.

3/II/5D Algebra and Geometry

Let $G = \langle g, h \mid h^2 = 1, g^6 = 1, hgh^{-1} = g^{-1} \rangle$ be the dihedral group of order 12.

i) List all the subgroups of G of order 2. Which of them are normal?

ii) Now list all the remaining proper subgroups of G. [There are 6+3 of them.]

iii) For each proper normal subgroup N of G, describe the quotient group G/N.

iv) Show that G is not isomorphic to the alternating group A_4.

3/II/6D Algebra and Geometry

State the conditions on a matrix A that ensure it represents a rotation of $\mathbb{R}^3$ with respect to the standard basis.

Check that the matrix

$$A = \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$

represents a rotation. Find its axis of rotation $\mathbf{n}$.

Let Π be the plane perpendicular to the axis $\mathbf{n}$. The matrix A induces a rotation of Π by an angle θ. Find $\cos\theta$.

3/II/7D Algebra and Geometry

Let A be a real symmetric matrix. Show that all the eigenvalues of A are real, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Give an example of a non-zero *complex* (2×2) symmetric matrix whose only eigenvalues are zero. Is it diagonalisable?

3/II/8D Algebra and Geometry

Compute the characteristic polynomial of

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4-s & 2s-2 \\ 0 & -2s+2 & 4s-1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A for all values of s.

For which values of s is A diagonalisable?

1/I/3D Analysis

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

State the Least Upper Bound Axiom for the real numbers.

Starting from the Least Upper Bound Axiom, show that if (a_n) is a bounded monotonic sequence of real numbers, then it converges.

1/I/4E Analysis

Let $f(x) = (1+x)^{1/2}$ for $x \in (-1,1)$. Show by induction or otherwise that for every integer $r \geq 1$,

$$f^{(r)}(x) = (-1)^{r-1} \frac{(2r-2)!}{2^{2r-1}(r-1)!}(1+x)^{\frac{1}{2}-r}.$$

Evaluate the series

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-2)!}{8^r r!(r-1)!}.$$

[You may use Taylor's Theorem in the form

$$f(x) = f(0) + \sum_{r=1}^{n} \frac{f^{(r)}(0)}{r!} x^r + \int_0^x \frac{(x-t)^n f^{(n+1)}(t)}{n!}\, dt$$

without proof.]

1/II/9D Analysis

i) State Rolle's theorem.

Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions which are differentiable on (a,b).

ii) Prove that for some $c \in (a,b)$,

$$(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c).$$

iii) Suppose that $f(a) = g(a) = 0$, and that $\lim_{x \to a+} \frac{f'(x)}{g'(x)}$ exists and is equal to L. Prove that $\lim_{x \to a+} \frac{f(x)}{g(x)}$ exists and is also equal to L.

[You may assume there exists a $\delta > 0$ such that, for all $x \in (a, a+\delta)$, $g'(x) \neq 0$ and $g(x) \neq 0$.]

iv) Evaluate $\lim_{x \to 0} \frac{\log \cos x}{x^2}$.

1/II/10E Analysis

Define, for an integer $n \geq 0$,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

Show that for every $n \geq 2$, $nI_n = (n-1)I_{n-2}$, and deduce that

$$I_{2n} = \frac{(2n)!}{(2^n n!)^2} \frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}.$$

Show that $0 < I_n < I_{n-1}$, and that

$$\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1.$$

Hence prove that

$$\lim_{n \to \infty} \frac{2^{4n+1}(n!)^4}{(2n+1)(2n)!^2} = \pi.$$

1/II/11F Analysis

Let f be defined on $\mathbb{R}$, and assume that there exists at least one point $x_0 \in \mathbb{R}$ at which f is continuous. Suppose also that, for every $x, y \in \mathbb{R}$, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Show that f is continuous on $\mathbb{R}$.

Show that there exists a constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Suppose that g is a continuous function defined on $\mathbb{R}$ and that, for every $x, y \in \mathbb{R}$, g satisfies the equation

$$g(x+y) = g(x)g(y).$$

Show that if g is not identically zero, then g is everywhere positive. Find the general form of g.

1/II/12F **Analysis**

(i) Show that if $a_n > 0$, $b_n > 0$ and

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$$

for all $n \geqslant 1$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Let

$$c_n = \binom{2n}{n} 4^{-n}.$$

By considering $\log c_n$, or otherwise, show that $c_n \to 0$ as $n \to \infty$.

[*Hint*: $\log(1-x) \leqslant -x$ for $x \in (0,1)$.]

(iii) Determine the convergence or otherwise of

$$\sum_{n=1}^{\infty} \binom{2n}{n} x^n$$

for (a) $x = \frac{1}{4}$, (b) $x = -\frac{1}{4}$.

2/I/1B **Differential Equations**

By writing $y(x) = mx$ where m is a constant, solve the differential equation

$$\frac{dy}{dx} = \frac{x - 2y}{2x + y}$$

and find the possible values of m.

Describe the isoclines of this differential equation and sketch the flow vectors. Use these to sketch at least two characteristically different solution curves.

Now, by making the substitution $y(x) = xu(x)$ or otherwise, find the solution of the differential equation which satisfies $y(0) = 1$.

2/I/2B **Differential Equations**

Find two linearly independent solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2 y = 0\,.$$

Find also the solution of

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2 y = e^{-px}$$

that satisfies

$$y = 0,\ \frac{dy}{dx} = 0 \quad \text{at } x = 0\,.$$

2/II/5B Differential Equations

Construct a series solution $y = y_1(x)$ valid in the neighbourhood of $x = 0$, for the differential equation

$$\frac{d^2y}{dx^2} + 4x^3\frac{dy}{dx} + x^2y = 0\,,$$

satisfying

$$y_1 = 1,\ \frac{dy_1}{dx} = 0 \quad \text{at } x = 0\,.$$

Find also a second solution $y = y_2(x)$ which satisfies

$$y_2 = 0,\ \frac{dy_2}{dx} = 1 \quad \text{at } x = 0\,.$$

Obtain an expression for the Wronskian of these two solutions and show that

$$y_2(x) = y_1(x)\int_0^x \frac{e^{-\xi^4}}{y_1^2(\xi)}\,d\xi\,.$$

2/II/6B Differential Equations

Two solutions of the recurrence relation

$$x_{n+2} + b(n)x_{n+1} + c(n)x_n = 0$$

are given as p_n and q_n, and their Wronskian is defined to be

$$W_n = p_n q_{n+1} - p_{n+1} q_n\,.$$

Show that

$$W_{n+1} = W_1 \prod_{m=1}^{n} c(m)\,. \qquad (*)$$

Suppose that $b(n) = \alpha$, where α is a real constant lying in the range $[-2, 2]$, and that $c(n) = 1$. Show that two solutions are $x_n = e^{in\theta}$ and $x_n = e^{-in\theta}$, where $\cos\theta = -\alpha/2$. Evaluate the Wronskian of these two solutions and verify $(*)$.

2/II/7B Differential Equations

Show how a second-order differential equation $\ddot{x} = f(x, \dot{x})$ may be transformed into a pair of coupled first-order equations. Explain what is meant by a *critical point* on the phase diagram for a pair of first-order equations. Hence find the critical points of the following equations. Describe their stability type, sketching their behaviour near the critical points on a phase diagram.

(i) $\ddot{x} + \cos x = 0$

(ii) $\ddot{x} + x(x^2 + \lambda x + 1) = 0, \quad$ for $\lambda = 1, \ 5/2\,.$

Sketch the phase portraits of these equations marking clearly the direction of flow.

2/II/8B Differential Equations

Construct the general solution of the system of equations

$$\begin{aligned} \dot{x} + 4x + 3y &= 0 \\ \dot{y} + 4y - 3x &= 0 \end{aligned}$$

in the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{x} = \sum_{j=1}^{2} a_j \mathbf{x}^{(j)} e^{\lambda_j t}$$

and find the eigenvectors $\mathbf{x}^{(j)}$ and eigenvalues λ_j.

Explain what is meant by resonance in a forced system of linear differential equations.

Consider the forced system

$$\begin{aligned} \dot{x} + 4x + 3y &= \sum_{j=1}^{2} p_j e^{\lambda_j t} \\ \dot{y} + 4y - 3x &= \sum_{j=1}^{2} q_j e^{\lambda_j t}\,. \end{aligned}$$

Find conditions on p_j and q_j $(j = 1, 2)$ such that there is no resonant response to the forcing.

4/I/3A **Dynamics**

A lecturer driving his car of mass m_1 along the flat at speed U_1 accidentally collides with a stationary vehicle of mass m_2. As both vehicles are old and very solidly built, neither suffers damage in the collision: they simply bounce elastically off each other in a straight line. Determine how both vehicles are moving after the collision if neither driver applied their brakes. State any assumptions made and consider all possible values of the mass ratio $R = m_1/m_2$. You may neglect friction and other such losses.

An undergraduate drives into a rigid rock wall at speed V. The undergraduate's car of mass M is modern and has a crumple zone of length L at its front. As this zone crumples upon impact, it exerts a net force $F = (L-y)^{-1/2}$ on the car, where y is the amount the zone has crumpled. Determine the value of y at the point the car stops moving forwards as a function of V, where $V < 2L^{\frac{1}{4}}/M^{\frac{1}{2}}$.

4/I/4A **Dynamics**

A small spherical bubble of radius a containing carbon dioxide rises in water due to a buoyancy force $\rho g V$, where ρ is the density of water, g is gravitational attraction and V is the volume of the bubble. The drag on a bubble moving at speed u is $6\pi\mu a u$, where μ is the dynamic viscosity of water, and an accelerating bubble acts like a particle of mass $\alpha\rho V$, for some constant α. Find the location at time t of a bubble released from rest at $t = 0$ and show the bubble approaches a steady rise speed

$$U = \frac{2}{9}\frac{\rho g}{\mu}a^2. \qquad (*)$$

Under some circumstances the carbon dioxide gradually dissolves in the water, which leads to the bubble radius varying as $a^2 = a_0^2 - \beta t$, where a_0 is the bubble radius at $t = 0$ and β is a constant. Under the assumption that the bubble rises at speed given by $(*)$, determine the height to which it rises before it disappears.

4/II/9A **Dynamics**

A horizontal table oscillates with a displacement $\mathbf{A}\sin\omega t$, where $\mathbf{A} = (A_x, 0, A_z)$ is the amplitude vector and ω the angular frequency in an inertial frame of reference with the z axis vertically upwards, normal to the table. A block sitting on the table has mass m and linear friction that results in a force $\mathbf{F} = -\lambda\mathbf{u}$, where λ is a constant and $\mathbf{u}$ is the velocity difference between the block and the table. Derive the equations of motion for this block in the frame of reference of the table using axes (ξ, η, ζ) on the table parallel to the axes (x, y, z) in the inertial frame.

For the case where $A_z = 0$, show that at late time the block will approach the steady orbit

$$\xi = \xi_0 - A_x \sin\theta \cos(\omega t - \theta),$$

where

$$\sin^2\theta = \frac{m^2\omega^2}{\lambda^2 + m^2\omega^2}$$

and ξ_0 is a constant.

Given that there are no attractive forces between block and table, show that the block will only remain in contact with the table if $\omega^2 A_z < g$.

4/II/10A **Dynamics**

A small probe of mass m is in low orbit about a planet of mass M. If there is no drag on the probe then its orbit is governed by

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^3}\,\mathbf{r},$$

where $\mathbf{r}$ is the location of the probe relative to the centre of the planet and G is the gravitational constant. Show that the basic orbital trajectory is elliptical. Determine the orbital period for the probe if it is in a circular orbit at a distance r_0 from the centre of the planet.

Data returned by the probe shows that the planet has a very extensive but diffuse atmosphere. This atmosphere induces a drag on the probe that may be approximated by the linear law $\mathbf{D} = -A\dot{\mathbf{r}}$, where $\mathbf{D}$ is the drag force and A is a constant. Show that the angular momentum of the probe about the planet decays exponentially.

4/II/11A Dynamics

A particle of mass m and charge q moves through a magnetic field $\mathbf{B}$. There is no electric field or external force so that the particle obeys

$$m\ddot{\mathbf{r}} = q\,\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}$ is the location of the particle. Prove that the kinetic energy of the particle is preserved.

Consider an axisymmetric magnetic field described by $\mathbf{B} = (0, 0, B(r))$ in cylindrical polar coordinates $\mathbf{r} = (r, \theta, z)$. Determine the angular velocity of a circular orbit centred on $\mathbf{r} = \mathbf{0}$.

For a general orbit when $B(r) = B_0/r$, show that the angular momentum about the z-axis varies as $L = L_0 - qB_0(r - r_0)$, where L_0 is the angular momentum at radius r_0. Determine and sketch the relationship between $\dot{r}^2$ and r. [Hint: Use conservation of energy.] What is the escape velocity for the particle?

4/II/12A Dynamics

A circular cylinder of radius a, length L and mass m is rolling along a surface. Show that its moment of inertia is given by $\frac{1}{2}ma^2$.

At $t = 0$ the cylinder is at the bottom of a slope making an angle α to the horizontal, and is rolling with velocity V and angular velocity V/a. Assuming slippage does not occur, determine the position of the cylinder as a function of time. What is the maximum height that the cylinder reaches?

The frictional force between the cylinder and surface is given by $\mu mg \cos\alpha$, where μ is the friction coefficient. Show that the cylinder begins to slip rather than roll if $\tan\alpha > 3\mu$. Determine as a function of time the location, speed and angular velocity of the cylinder on the slope if this condition is satisfied. Show that slipping continues as the cylinder ascends and descends the slope. Find also the maximum height the cylinder reaches, and its speed and angular velocity when it returns to the bottom of the slope.

4/I/1E Numbers and Sets

(a) Use Euclid's algorithm to find positive integers m, n such that $79m - 100n = 1$.

(b) Determine all integer solutions of the congruence

$$237x \equiv 21 \pmod{300}.$$

(c) Find the set of all integers x satisfying the simultaneous congruences

$$x \equiv 8 \pmod{79}$$
$$x \equiv 11 \pmod{100}.$$

4/I/2E Numbers and Sets

Prove by induction the following statements:

i) For every integer $n \geq 1$,

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n).$$

ii) For every integer $n \geq 1$, $n^3 + 5n$ is divisible by 6.

4/II/5E Numbers and Sets

Show that the set of all subsets of $\mathbb{N}$ is uncountable, and that the set of all finite subsets of $\mathbb{N}$ is countable.

Let X be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$, and let $Y \subset X$ be the set

$$Y = \{f \in X \mid \text{for all but finitely many } n \in \mathbb{N},\ f(n) = n\}.$$

Show that X is uncountable, but that Y is countable.

4/II/6E **Numbers and Sets**

Prove Fermat's Theorem: if p is prime and $(x,p)=1$ then $x^{p-1} \equiv 1 \pmod{p}$.

Let n and x be positive integers with $(x,n)=1$. Show that if $n=mp$ where p is prime and $(m,p)=1$, then

$$x^{n-1} \equiv 1 \pmod{p} \quad \text{if and only if} \quad x^{m-1} \equiv 1 \pmod{p}.$$

Now assume that n is a product of distinct primes. Show that $x^{n-1} \equiv 1 \pmod{n}$ if and only if, for every prime divisor p of n,

$$x^{(n/p)-1} \equiv 1 \pmod{p}.$$

Deduce that if every prime divisor p of n satisfies $(p-1)|(n-1)$, then for every x with $(x,n)=1$, the congruence

$$x^{n-1} \equiv 1 \pmod{n}$$

holds.

4/II/7E **Numbers and Sets**

Polynomials $P_r(X)$ for $r \geq 0$ are defined by

$$P_0(X) = 1$$
$$P_r(X) = \frac{X(X-1)\cdots(X-r+1)}{r!} = \prod_{i=1}^{r} \frac{X-i+1}{i} \qquad \text{for } r \geq 1.$$

Show that $P_r(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, and that if $r \geq 1$ then $P_r(X) - P_r(X-1) = P_{r-1}(X-1)$.

Prove that if F is any polynomial of degree d with rational coefficients, then there are unique rational numbers $c_r(F)$ $(0 \leq r \leq d)$ for which

$$F(X) = \sum_{r=0}^{d} c_r(F) P_r(X).$$

Let $\Delta F(X) = F(X+1) - F(X)$. Show that

$$\Delta F(X) = \sum_{r=0}^{d-1} c_{r+1}(F) P_r(X).$$

Show also that, if F and G are polynomials such that $\Delta F = \Delta G$, then $F - G$ is a constant.

By induction on the degree of F, or otherwise, show that if $F(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, then $c_r(F) \in \mathbb{Z}$ for all r.

4/II/8E Numbers and Sets

Let X be a finite set, $X_1, \ldots, X_m$ subsets of X and $Y = X \setminus \bigcup X_i$. Let g_i be the characteristic function of X_i, so that

$$g_i(x) = \begin{cases} 1 & \text{if } x \in X_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $f\colon X \to \mathbb{R}$ be any function. By considering the expression

$$\sum_{x \in X} f(x) \prod_{i=1}^{m} (1 - g_i(x)),$$

or otherwise, prove the Inclusion–Exclusion Principle in the form

$$\sum_{x \in Y} f(x) = \sum_{r \geq 0} (-1)^r \sum_{i_1 < \cdots < i_r} \Bigg(\sum_{x \in X_{i_1} \cap \cdots \cap X_{i_r}} f(x) \Bigg).$$

Let $n > 1$ be an integer. For an integer m dividing n let

$$X_m = \{0 \leq x < n \mid x \equiv 0 \pmod{m}\}.$$

By considering the sets X_p for prime divisors p of n, show that

$$\phi(n) = n \prod_{p|n} \Big(1 - \frac{1}{p}\Big)$$

(where ϕ is Euler's function) and

$$\sum_{\substack{0 < x < n \\ (x,n)=1}} x = \frac{n^2}{2} \prod_{p|n} \Big(1 - \frac{1}{p}\Big).$$

2/I/3F **Probability**

Define the covariance, $\text{cov}(X, Y)$, of two random variables X and Y.

Prove, or give a counterexample to, each of the following statements.

(a) For any random variables X, Y, Z

$$\text{cov}(X + Y, Z) = \text{cov}(X, Z) + \text{cov}(Y, Z).$$

(b) If X and Y are identically distributed, not necessarily independent, random variables then

$$\text{cov}(X + Y, X - Y) = 0.$$

2/I/4F **Probability**

The random variable X has probability density function

$$f(x) = \begin{cases} cx(1-x) & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine c, and the mean and variance of X.

2/II/9F **Probability**

Let X be a positive-integer valued random variable. Define its *probability generating function* p_X. Show that if X and Y are independent positive-integer valued random variables, then $p_{X+Y} = p_X p_Y$.

A non-standard pair of dice is a pair of six-sided unbiased dice whose faces are numbered with strictly positive integers in a non-standard way (for example, $(2, 2, 2, 3, 5, 7)$ and $(1, 1, 5, 6, 7, 8)$). Show that there exists a non-standard pair of dice A and B such that when thrown

$$P\{\text{total shown by } A \text{ and } B \text{ is } n\} = P\{\text{total shown by pair of ordinary dice is } n\}$$

for all $2 \leqslant n \leqslant 12$.

[*Hint:* $(x + x^2 + x^3 + x^4 + x^5 + x^6) = x(1 + x)(1 + x^2 + x^4) = x(1 + x + x^2)(1 + x^3)$.]

2/II/10F **Probability**

Define the *conditional probability* $P(A \mid B)$ of the event A given the event B.

A bag contains four coins, each of which when tossed is equally likely to land on either of its two faces. One of the coins shows a head on each of its two sides, while each of the other three coins shows a head on only one side. A coin is chosen at random, and tossed three times in succession. If heads turn up each time, what is the probability that if the coin is tossed once more it will turn up heads again? Describe the sample space you use and explain carefully your calculations.

2/II/11F **Probability**

The random variables X_1 and X_2 are independent, and each has an exponential distribution with parameter λ. Find the joint density function of

$$Y_1 = X_1 + X_2\,, \quad Y_2 = X_1/X_2\,,$$

and show that Y_1 and Y_2 are independent. What is the density of Y_2?

2/II/12F **Probability**

Let $A_1, A_2, \ldots, A_r$ be events such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Show that the number N of events that occur satisfies

$$P(N=0) \quad = \quad 1 - \sum_{i=1}^{r} P(A_i)\,.$$

Planet Zog is a sphere with centre O. A number N of spaceships land at random on its surface, their positions being independent, each uniformly distributed over the surface. A spaceship at A is in direct radio contact with another point B on the surface if $\angle AOB < \frac{\pi}{2}$. Calculate the probability that every point on the surface of the planet is in direct radio contact with at least one of the N spaceships.

[*Hint*: The intersection of the surface of a sphere with a plane through the centre of the sphere is called a *great circle*. You may find it helpful to use the fact that N random great circles partition the surface of a sphere into $N(N-1)+2$ disjoint regions with probability one.]

3/I/3C **Vector Calculus**

If $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields, show that

(i) $\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - \mathbf{G}(\boldsymbol{\nabla} \cdot \mathbf{F}) + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}\,,$

(ii) $\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F})\,.$

3/I/4C **Vector Calculus**

Define the curvature, κ, of a curve in $\mathbb{R}^3$.

The curve C is parametrised by

$$\mathbf{x}(t) = \left(\frac{1}{2}e^t \cos t,\ \frac{1}{2}e^t \sin t,\ \frac{1}{\sqrt{2}}e^t\right) \quad \text{for } -\infty < t < \infty\,.$$

Obtain a parametrisation of the curve in terms of its arc length, s, measured from the origin. Hence obtain its curvature, $\kappa(s)$, as a function of s.

3/II/9C **Vector Calculus**

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ state if the following implications are true or false. (No justification is required.)

(i) f is differentiable $\Rightarrow$ f is continuous.

(ii) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist $\Rightarrow$ f is continuous.

(iii) directional derivatives $\dfrac{\partial f}{\partial \mathbf{n}}$ exist for all unit vectors $\mathbf{n} \in \mathbb{R}^2 \Rightarrow f$ is differentiable.

(iv) f is differentiable $\Rightarrow$ $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous.

(v) all second order partial derivatives of f exist $\Rightarrow \dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$.

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that f is continuous at $(0,0)$ and find the partial derivatives $\dfrac{\partial f}{\partial x}(0,y)$ and $\dfrac{\partial f}{\partial y}(x,0)$. Then show that f is differentiable at $(0,0)$ and find its derivative. Investigate whether the second order partial derivatives $\dfrac{\partial^2 f}{\partial x\,\partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}(0,0)$ are the same. Are the second order partial derivatives of f at $(0,0)$ continuous? Justify your answer.

3/II/10C Vector Calculus

Explain what is meant by an exact differential. The three-dimensional vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = \left(e^x z^3 + 3x^2(e^y - e^z),\, e^y(x^3 - z^3),\, 3z^2(e^x - e^y) - e^z x^3\right).$$

Find the most general function that has $\mathbf{F} \cdot \mathbf{dx}$ as its differential.

Hence show that the line integral

$$\int_{P_1}^{P_2} \mathbf{F} \cdot \mathbf{dx}$$

along any path in $\mathbb{R}^3$ between points $P_1 = (0, a, 0)$ and $P_2 = (b, b, b)$ vanishes for any values of a and b.

The two-dimensional vector field $\mathbf{G}$ is defined at all points in $\mathbb{R}^2$ except $(0, 0)$ by

$$\mathbf{G} = \left(\frac{-y}{x^2 + y^2},\, \frac{x}{x^2 + y^2}\right).$$

($\mathbf{G}$ is not defined at $(0, 0)$.) Show that

$$\oint_C \mathbf{G} \cdot \mathbf{dx} = 2\pi$$

for any closed curve C in $\mathbb{R}^2$ that goes around $(0, 0)$ anticlockwise precisely once without passing through $(0, 0)$.

3/II/11C Vector Calculus

Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0, 0, 0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, 0, 0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(\frac{-1}{4}, 0, 0)$. The eccentrically shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1. Is there a point in the interior of S_3 where a test particle would remain stably at rest? Justify your answer.

3/II/12C **Vector Calculus**

State (without proof) the divergence theorem for a vector field $\mathbf{F}$ with continuous first-order partial derivatives throughout a volume V enclosed by a bounded oriented piecewise-smooth non-self-intersecting surface S.

By calculating the relevant volume and surface integrals explicitly, verify the divergence theorem for the vector field

$$\mathbf{F} = \left(x^3 + 2xy^2,\ y^3 + 2yz^2,\ z^3 + 2zx^2\right),$$

defined within a sphere of radius R centred at the origin.

Suppose that functions ϕ, ψ are continuous and that their first and second partial derivatives are all also continuous in a region V bounded by a smooth surface S.

Show that

$$(1) \qquad \int_V (\phi \nabla^2 \psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS}.$$

$$(2) \qquad \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS} - \int_S \psi \boldsymbol{\nabla}\phi \cdot \mathbf{dS}.$$

Hence show that if $\rho(\mathbf{x})$ is a continuous function on V and $g(\mathbf{x})$ a continuous function on S and ϕ_1 and ϕ_2 are two continuous functions such that

$$\nabla^2 \phi_1(\mathbf{x}) = \nabla^2 \phi_2(\mathbf{x}) = \rho(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ in } V, \text{ and}$$
$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ on } S,$$

then $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x}$ in V.