UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Wednesday 2 June 2004 9 to 12

PAPER 2

Before you begin read these instructions carefully.

The number of marks for each question is the same.

Additional credit will be given for a substantially complete answer.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **1B, 17B** *should be in one bundle and* **14J, 16J** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B Principles of Dynamics

(i) Consider a light rigid circular wire of radius a and centre O. The wire lies in a vertical plane, which rotates about the vertical axis through O. At time t the plane containing the wire makes an angle $\phi(t)$ with a fixed vertical plane. A bead of mass m is threaded onto the wire. The bead slides without friction along the wire, and its location is denoted by A. The angle between the line OA and the downward vertical is $\theta(t)$.

Show that the Lagrangian of the system is

$$\frac{ma^2}{2}\dot{\theta}^2 + \frac{ma^2}{2}\dot{\phi}^2 \sin^2\theta + mga\cos\theta \ .$$

Calculate two independent constants of the motion, and explain their physical significance.

(ii) A dynamical system has Hamiltonian $H(q,p,\lambda)$, where λ is a parameter. Consider an ensemble of identical systems chosen so that the number density of systems, $f(q,p,t)$, in the phase space element $dq\,dp$ is either zero or one. Prove Liouville's Theorem, namely that the total area of phase space occupied by the ensemble is time-independent.

Now consider a single system undergoing periodic motion $q(t), p(t)$. Give a heuristic argument based on Liouville's Theorem to show that the area enclosed by the orbit,

$$I = \oint p\,dq\,,$$

is approximately conserved as the parameter λ is slowly varied (i.e. that I is an adiabatic invariant).

Consider $H(q,p,\lambda) = \frac{1}{2}p^2 + \lambda q^{2n}$, with n a positive integer. Show that as λ is slowly varied the energy of the system, E, varies as

$$E \propto \lambda^{1/(n+1)}\,.$$

2F Functional Analysis

(i) Prove Riesz's Lemma, that if V is a normed space and A is a vector subspace of V such that for some $0 \leqslant k < 1$ we have $d(x, A) \leqslant k$ for all $x \in V$ with $||x|| = 1$, then A is dense in V. [Here $d(x, A)$ denotes the distance from x to A.]

Deduce that any normed space whose unit ball is compact is finite-dimensional. [*You may assume that every finite-dimensional normed space is complete.*]

Give an example of a sequence $f_1, f_2, \ldots$ in an infinite-dimensional normed space such that $||f_n|| \leqslant 1$ for all n, but $f_1, f_2, \ldots$ has no convergent subsequence.

(ii) Let V be a vector space, and let $||.||_1$ and $||.||_2$ be two norms on V. What does it mean to say that $||.||_1$ and $||.||_2$ are *equivalent*?

Show that on a finite-dimensional vector space all norms are equivalent. Deduce that every finite-dimensional normed space is complete.

Exhibit two norms on the vector space l^1 that are not equivalent.

In addition, exhibit two norms on the vector space l^∞ that are not equivalent.

3G Groups, Rings and Fields

(i) State Gauss' Lemma on polynomial irreducibility. State and prove Eisenstein's criterion.

(ii) Which of the following polynomials are irreducible over $\mathbb{Q}$? Justify your answers.

(a) $x^7 - 3x^3 + 18x + 12$

(b) $x^4 - 4x^3 + 11x^2 - 3x - 5$

(c) $1 + x + x^2 + \ldots + x^{p-1}$ with p prime

[*Hint: consider substituting* $y = x - 1$.]

(d) $x^n + px + p^2$ with p prime.

[*Hint: show any factor has degree at least two, and consider powers of p dividing coefficients.*]

4B Dynamics of Differential Equations

(i) Define carefully what is meant by a *Hopf bifurcation* in a two-dimensional dynamical system. Write down the normal form for this bifurcation, correct to cubic order, and distinguish between bifurcations of supercritical and subcritical type. Describe, without detailed calculations, how a general two-dimensional system with a Hopf bifurcation at the origin can be reduced to normal form by a near-identity transformation.

(ii) A *Takens-Bogdanov bifurcation* of a fixed point of a two-dimensional system is characterised by a Jacobian with the canonical form

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

at the bifurcation point. Consider the system

$$\begin{aligned} \dot{x} &= y + \alpha_1 x^2 + \beta_1 xy + \gamma_1 y^2 \\ \dot{y} &= \quad\ \ \alpha_2 x^2 + \beta_2 xy + \gamma_2 y^2 \ . \end{aligned}$$

Show that a near-identity transformation of the form

$$\begin{aligned} \xi &= x + a_1 x^2 + b_1 xy + c_1 y^2 \\ \eta &= y + a_2 x^2 + b_2 xy + c_2 y^2 \end{aligned}$$

exists that reduces the system to the normal (canonical) form, correct up to quadratic terms,

$$\dot{\xi} = \eta, \quad \dot{\eta} = \alpha_2 \xi^2 + (\beta_2 + 2\alpha_1)\xi\eta.$$

It is known that the general form of the equations *near* the bifurcation point can be written (setting $p = \alpha_2$, $q = \beta_2 + 2\alpha_1$)

$$\dot{\xi} = \eta, \quad \dot{\eta} = \lambda\xi + \mu\eta + p\xi^2 + q\xi\eta.$$

Find all the fixed points of this system, and the values of λ, μ for which these fixed points have (a) steady state bifurcations and (b) Hopf bifurcations.

5F Combinatorics

State and prove Sperner's lemma on antichains.

The family $\mathcal{A} \subset \mathcal{P}[n]$ is said to *split* $[n]$ if, for all distinct $i, j \in [n]$, there exists $A \in \mathcal{A}$ with $i \in A$ but $j \notin A$. Prove that if $\mathcal{A}$ splits $[n]$ then $n \leq \binom{a}{\lfloor a/2 \rfloor}$, where $a = |\mathcal{A}|$.

Show moreover that, if $\mathcal{A}$ splits $[n]$ and no element of $[n]$ is in more than $k < \lfloor a/2 \rfloor$ members of $\mathcal{A}$, then $n \leq \binom{a}{k}$.

6G Representation Theory

Let H be a group with three generators c, g, h and relations $c^p = g^p = h^p = 1$, $cg = gc$, $ch = hc$ and $gh = chg$ where p is a prime number.

(a) Show that $|H| = p^3$. Show that the conjugacy classes of H are the singletons $\{1\}, \{c\}, \ldots, \{c^{p-1}\}$ and the sets $\{g^m h^n, cg^m h^n, \ldots, c^{p-1} g^m h^n\}$, as m, n range from 0 to $p-1$, but $(m,n) \neq (0,0)$.

(b) Find p^2 1-dimensional representations of H.

(c) Let $\omega \neq 1$ be a pth root of unity. Show that the following defines an *irreducible* representation of H on $\mathbb{C}^p$:

$$\begin{aligned} \rho(c) &= \omega \mathrm{Id}, \\ \rho(g)\mathbf{e}_k &= \omega^k \mathbf{e}_k, \\ \rho(h)\mathbf{e}_p &= \mathbf{e}_1 \text{ and } \rho(h)\mathbf{e}_k = \mathbf{e}_{k+1} \text{ if } \ k < p \end{aligned}$$

where the $\mathbf{e}_k$ are the standard basis vectors of $\mathbb{C}^p$.

(d) Show that (b) and (c) cover all irreducible isomorphism classes.

7H Differentiable Manifolds

For each of the following assertions, either provide a proof or give and justify a counterexample.

[*You may use, without proof, your knowledge of the de Rham cohomology of surfaces.*]

(a) A smooth map $f : S^2 \to T^2$ must have degree zero.

(b) An embedding $\varphi : S^1 \to \Sigma_g$ extends to an embedding $\bar{\varphi} : D^2 \to \Sigma_g$ if and only if the map

$$\int_{\varphi(S^1)} : \ H^1(\Sigma_g) \to \mathbb{R}$$

is the zero map.

(c) $\mathbb{RP}^1 \times \mathbb{RP}^2$ is orientable.

(d) The surface Σ_g admits the structure of a Lie group if and only if $g = 1$.

8G Algebraic Topology

Let K and L be finite simplicial complexes. Define the n-th chain group $C_n(K)$ and the boundary homomorphism $d_n : C_n(K) \to C_{n-1}(K)$. Prove that $d_{n-1}d_n = 0$ and define the homology groups of K. Explain briefly how a simplicial map $f : K \to L$ induces a homomorphism $f_\star$ of homology groups.

Suppose now that K consists of the *proper* faces of a 3-dimensional simplex. Calculate from first principles the homology groups of K. If a simplicial map $f : K \to K$ gives a homeomorphism of the underlying polyhedron $|K|$, is the induced homology map $f_\star$ necessarily the identity?

9H Number Fields

Let m be an integer greater than 1 and let ζ_m denote a primitive m-th root of unity in $\mathbb{C}$. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\zeta_m)$. If p is a prime number with $(p, m) = 1$, outline the proof that

$$p\mathcal{O} = \wp_1 \ldots \wp_r,$$

where the $\wp_i$ are distinct prime ideals of $\mathcal{O}$, and $r = \varphi(m)/f$ with f the least integer $\geqslant 1$ such that $p^f \equiv 1 \bmod m$. [Here $\varphi(m)$ is the Euler φ-function of m].

Determine the factorisations of $2, 3, 5$ and 11 in $\mathbb{Q}(\zeta_5)$. For each integer $n \geqslant 1$, prove that, in the ring of integers of $\mathbb{Q}(\zeta_{5^n})$, there is a unique prime ideal dividing 2, and a unique prime ideal dividing 3.

10H Algebraic Curves

For each of the following curves C

(i) $C = \{(x, y) \in \mathbb{A}^2 | x^3 - x = y^2\}$ (ii) $C = \{(x, y) \in \mathbb{A}^2 | x^2y + xy^2 = x^4 + y^4\}$

compute the points at infinity of $\bar{C} \subset \mathbb{P}^2$ (i.e. describe $\bar{C} \setminus C$), and find the singular points of the projective curve $\bar{C}$.

At which points of $\bar{C}$ is the rational map $\bar{C} \dashrightarrow \mathbb{P}^1$, given by $(X : Y : Z) \mapsto (X : Y)$, not defined? Justify your answer.

11F Logic, Computation and Set Theory

Define the sets V_α, $\alpha \in ON$. Show that each V_α is transitive, and explain why $V_\alpha \subseteq V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

Which of the following are true and which are false? Give proofs or counterexamples as appropriate. You may assume standard properties of rank.

(a) If the rank of a set x is a (non-zero) limit then x is infinite.

(b) If the rank of a set x is a successor then x is finite.

(c) If the rank of a set x is countable then x is countable.

12I Probability and Measure

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $1 \leqslant p \leqslant \infty$.

(a) Define the L^p-norm $||f||_p$ of a measurable function $f : \Omega \to \mathbb{R}$, and define the space $L^p(\Omega, \mathcal{F}, \mu)$.

(b) Prove Minkowski's inequality:

$$||f+g||_p \leqslant ||f||_p + ||g||_p \quad \text{for} \quad f, g \in L^p(\Omega, \mathcal{F}, \mu),\ 1 \leqslant p \leqslant \infty.$$

[*You may use Hölder's inequality without proof provided it is clearly stated.*]

(c) Explain what is meant by saying that $L^p(\Omega, \mathcal{F}, \mu)$ is *complete.* Show that $L^\infty(\Omega, \mathcal{F}, \mu)$ is complete.

(d) Suppose that $\{f_n : n \geqslant 1\}$ is a sequence of measurable functions satisfying $||f_n||_p \to 0$ as $n \to \infty$.

(i) Show that if $p = \infty$, then $f_n \to 0$ almost everywhere.

(ii) When $1 \leqslant p < \infty$, give an example of a measure space $(\Omega, \mathcal{F}, \mu)$ and such a sequence $\{f_n\}$ such that, for all $\omega \in \Omega$, $f_n(\omega) \not\to 0$ as $n \to \infty$.

13I Applied Probability

Let M be a Poisson random measure of intensity λ on the plane $\mathbb{R}^2$. Denote by $C(r)$ the circle $\{x \in \mathbb{R}^2 : ||x|| < r\}$ of radius r in $\mathbb{R}^2$ centred at the origin and let R_k be the largest radius such that $C(R_k)$ contains precisely k points of M. [Thus $C(R_0)$ is the largest circle about the origin containing no points of M, $C(R_1)$ is the largest circle about the origin containing a single point of M, and so on.] Calculate $\mathbb{E}R_0$, $\mathbb{E}R_1$ and $\mathbb{E}R_2$.

Now let N be a Poisson random measure of intensity λ on the line $\mathbb{R}^1$. Let L_k be the length of the largest open interval that covers the origin and contains precisely k points of N. [Thus L_0 gives the length of the largest interval containing the origin but no points of N, L_1 gives the length of the largest interval containing the origin and a single point of N, and so on.] Calculate $\mathbb{E}L_0$, $\mathbb{E}L_1$ and $\mathbb{E}L_2$.

14J Information Theory

For integer-valued random variables X and Y, define the relative entropy $h_Y(X)$ of X relative to Y.

Prove that $h_Y(X) \geqslant 0$, with equality if and only if $\mathbb{P}(X = x) = \mathbb{P}(Y = x)$ for all x.

By considering Y, a geometric random variable with parameter chosen appropriately, show that if the mean $\mathbb{E}X = \mu < \infty$, then

$$h(X) \leqslant (\mu + 1)\log(\mu + 1) - \mu \log \mu \,,$$

with equality if X is geometric.

15I Optimization and Control

A gambler is presented with a sequence of $n \geqslant 6$ random numbers, $N_1, N_2, \ldots, N_n$, one at a time. The distribution of N_k is

$$P(N_k = k) = 1 - P(N_k = -k) = p \,,$$

where $1/(n-2) < p \leq 1/3$. The gambler must choose exactly one of the numbers, just after it has been presented and before any further numbers are presented, but must wait until all the numbers are presented before his payback can be decided. It costs £1 to play the game. The gambler receives payback as follows: nothing if he chooses the smallest of all the numbers, £2 if he chooses the largest of all the numbers, and £1 otherwise.

Show that there is an optimal strategy of the form "Choose the first number k such that either (i) $N_k > 0$ and $k \geq n - r_0$, or (ii) $k = n - 1$", where you should determine the constant r_0 as explicitly as you can.

16J Principles of Statistics

(i) In the context of a decision-theoretic approach to statistics, what is a *loss function*? a *decision rule*? the *risk function* of a decision rule? the *Bayes risk* of a decision rule? the *Bayes rule* with respect to a given prior distribution?

Show how the Bayes rule with respect to a given prior distribution is computed.

(ii) A sample of n people is to be tested for the presence of a certain condition. A single real-valued observation is made on each one; this observation comes from density f_0 if the condition is absent, and from density f_1 if the condition is present. Suppose $\theta_i = 0$ if the i^{th} person does not have the condition, $\theta_i = 1$ otherwise, and suppose that the prior distribution for the θ_i is that they are independent with common distribution $P(\theta_i = 1) = p \in (0,1)$, where p is known. If X_i denotes the observation made on the i^{th} person, what is the posterior distribution of the θ_i?

Now suppose that the loss function is defined by

$$L_0(\theta, a) \equiv \sum_{j=1}^{n} (\alpha a_j (1 - \theta_j) + \beta (1 - a_j) \theta_j)$$

for action $a \in [0,1]^n$, where α, β are positive constants. If π_j denotes the posterior probability that $\theta_j = 1$ given the data, prove that the Bayes rule for this prior and this loss function is to take $a_j = 1$ if π_j exceeds the threshold value $\alpha/(\alpha + \beta)$, and otherwise to take $a_j = 0$.

In an attempt to control the proportion of false positives, it is proposed to use a different loss function, namely,

$$L_1(\theta, a) \equiv L_0(\theta, a) + \gamma I_{\{\sum a_j > 0\}} \left(1 - \frac{\sum \theta_j a_j}{\sum a_j} \right),$$

where $\gamma > 0$. Prove that the Bayes rule is once again a threshold rule, that is, we take action $a_j = 1$ if and only if $\pi_j > \lambda$, and determine λ as fully as you can.

17B Nonlinear Dynamical Systems

(i) A linear system in $\mathbb{R}^2$ takes the form $\dot{\mathbf{x}} = \mathsf{A}\mathbf{x}$. Explain (without detailed calculation but by giving examples) how to classify the dynamics of the system in terms of the determinant and the trace of A. Show your classification graphically, and describe the dynamics that occurs on the boundaries of the different regions on your diagram.

(ii) A nonlinear system in $\mathbb{R}^2$ has the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f}(0) = 0$. The Jacobian (linearization) A of $\mathbf{f}$ at the origin is non-hyperbolic, with one eigenvalue of A in the left-hand half-plane. Define the *centre manifold* for this system, and explain (stating carefully any results you use) how the dynamics near the origin may be reduced to a one-dimensional system on the centre manifold.

A dynamical system of this type has the form

$$\begin{aligned}\dot{x} &= ax^3 + bxy + cx^5 + dx^3y + exy^2 + fx^7 + gx^5y \\ \dot{y} &= -y + x^2 - x^4\end{aligned}$$

Find the coefficients for the expansion of the centre manifold correct up to and including terms of order x^6, and write down in terms of these coefficients the equation for the dynamics on the centre manifold up to order x^7. Using this reduced equation, give a complete set of conditions on the coefficients $a, b, c, \ldots$ that guarantee that the origin is stable.

18D Partial Differential Equations

(a) State and prove the Duhamel principle for the wave equation.

(b) Let $u \in C^2([0,T] \times \mathbb{R}^n)$ be a solution of

$$u_{tt} + u_t - \Delta u + u = 0$$

where Δ is taken in the variables $x \in \mathbb{R}^n$ and $u_t = \partial_t u$ etc.

Using an 'energy method', or otherwise, show that, if $u = u_t = 0$ on the set $\{t = 0, |x - x_0| \leqslant t_0\}$ for some $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$, then u vanishes on the region $K(t,x) = \{(t,x) : 0 \leqslant t \leqslant t_0, |x - x_0| \leqslant t_0 - t\}$. Hence deduce uniqueness for the Cauchy problem for the above PDE with Schwartz initial data.

19D Methods of Mathematical Physics

(a) The Beta function is defined by

$$\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx \ .$$

Show that

$$\mathrm{B}(p,q) = \int_1^\infty x^{-p-q}(x-1)^{q-1}dx \ .$$

(b) The function $J(p,q)$ is defined by

$$J(p,q) = \int_\gamma t^{p-1}(1-t)^{q-1}dt \ ,$$

where the integrand has a branch cut along the positive real axis. Just above the cut, arg $t = 0$. For $t > 1$ just above the cut, arg $(1-t) = -\pi$. The contour γ runs from $t = \infty e^{2\pi i}$, round the origin in the negative sense, to $t = \infty$ (i.e. the contour is a reflection of the usual Hankel contour). What restriction must be placed on p and q for the integral to converge?

By evaluating $J(p,q)$ in two ways, show that

$$\left(1 - e^{2\pi i p}\right)\mathrm{B}(p,q) + \left(e^{-\pi i(q-1)} - e^{\pi i(2p+q-1)}\right)\mathrm{B}(1-p-q,q) = 0 \ ,$$

where p and q are any non-integer complex numbers.

Using the identity

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \ ,$$

deduce that

$$\Gamma(p)\Gamma(1-p)\sin(\pi p) = \Gamma(p+q)\Gamma(1-p-q)\sin[\pi(1-p-q)] \ ,$$

and hence that

$$\pi = \Gamma(q)\Gamma(1-q)\sin[\pi(1-q)] \ .$$

20D Numerical Analysis

(i) The *five-point equations*, which are obtained when the Poisson equation $\nabla^2 u = f$ (with Dirichlet boundary conditions) is discretized in a square, are

$$-u_{m-1,n} - u_{m,n-1} - u_{m+1,n} - u_{m,n+1} + 4u_{m,n} = f_{m,n}, \quad m, n = 1, 2, \ldots, M,$$

where $u_{0,n},\ u_{M+1,n},\ u_{m,0},\ u_{m,M+1} = 0$ for all $m, n = 1, 2, \ldots, M$.

Formulate the Gauss–Seidel method for the above linear system and prove its convergence. In the proof you should carefully state any theorems you use. [*You may use Part (ii) of this question.*]

(ii) By arranging the two-dimensional arrays $\{u_{m,n}\}_{m,n=1,\ldots,M}$ and $\{b_{m,n}\}_{m,n=1,\ldots,M}$ into the column vectors $\mathbf{u} \in \mathbb{R}^{M^2}$ and $\mathbf{b} \in \mathbb{R}^{M^2}$ respectively, the linear system described in Part (i) takes the matrix form $A\mathbf{u} = \mathbf{b}$. Prove that, regardless of the ordering of the points on the grid, the matrix A is symmetric and positive definite.

21C Electrodynamics

A particle of rest mass m and charge q moves along a path $x^a(s)$, where s is the particle's proper time. The equation of motion is

$$m\ddot{x}^a = qF^{ab}\eta_{bc}\dot{x}^c,$$

where $\dot{x}^a = dx^a/ds$ etc., F^{ab} is the Maxwell field tensor ($F^{01} = -E_x$, $F^{23} = -B_x$, where E_x and B_x are the x-components of the electric and magnetic fields) and η_{bc} is the Minkowski metric tensor. Show that $\dot{x}_a\ddot{x}^a = 0$ and interpret both the equation of motion and this equation in the classical limit.

The electromagnetic field is given in cartesian coordinates by $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, E)$, where E is constant and uniform. The particle starts from rest at the origin. Show that the orbit is given by

$$9x^2 = 2\alpha y^3, \quad z = 0,$$

where $\alpha = qE/m$.

22E Foundations of Quantum Mechanics

(i) The creation and annihilation operators for a harmonic oscillator of angular frequency ω satisfy the commutation relation $[a, a^\dagger] = 1$. Write down an expression for the Hamiltonian H in terms of a and $a^\dagger$.

There exists a unique ground state $|0\rangle$ of H such that $a|0\rangle = 0$. Explain how the space of eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$ of H is formed, and deduce the eigenenergies for these states. Show that

$$a|n\rangle \;=\; \sqrt{n}|n-1\rangle\ , \qquad a^\dagger|n\rangle \;=\; \sqrt{n+1}|n+1\rangle\ .$$

(ii) Write down the number operator N of the harmonic oscillator in terms of a and $a^\dagger$. Show that

$$N|n\rangle \;=\; n|n\rangle\ .$$

The operator K_r is defined to be

$$K_r \;=\; \frac{a^{\dagger\, r} a^r}{r!}\ , \qquad r = 0, 1, 2, \ldots .$$

Show that K_r commutes with N. Show also that

$$K_r|n\rangle = \begin{cases} \frac{n!}{(n-r)!\,r!}|n\rangle & r \le n\ , \\ 0 & r > n\ . \end{cases}$$

By considering the action of K_r on the state $|n\rangle$ show that

$$\sum_{r=0}^{\infty} (-1)^r K_r \;=\; |0\rangle\langle 0|\ .$$

23E Applications of Quantum Mechanics

The wave function for a single particle with a potential $V(r)$ has the asymptotic form for large r

$$\psi(r,\theta) \sim e^{ikr\cos\theta} + f(\theta)\,\frac{e^{ikr}}{r}\,.$$

How is $f(\theta)$ related to observable quantities? Show how $f(\theta)$ can be expressed in terms of phase shifts $\delta_\ell(k)$ for $\ell = 0, 1, 2, \ldots$.

Assume that $V(r) = 0$ for $r \geq a$, and let $R_\ell(r)$ denote the solution of the radial Schrödinger equation, regular at $r = 0$, with energy $\hbar^2 k^2/2m$ and angular momentum ℓ. Let $N_\ell(k) = aR_\ell{}'(a)/R_\ell(a)$. Show that

$$\tan\delta_\ell(k) = \frac{N_\ell(k)\; j_\ell(ka) - ka\, j_\ell{}'(ka)}{N_\ell(k)\; n_\ell(ka) - ka\, n_\ell{}'(ka)}\,.$$

Assuming that $N_\ell(k)$ is a smooth function for $k \approx 0$, determine the expected behaviour of $\delta_\ell(k)$ as $k \to 0$. Show that for $k \to 0$ then $f(\theta) \to c$, with c a constant, and determine c in terms of $N_0(0)$.

[*For $V = 0$ the two independent solutions of the radial Schrödinger equation are $j_\ell(kr)$ and $n_\ell(kr)$ with*

$$\begin{aligned}
j_\ell(\rho) &\sim \frac{1}{\rho}\sin(\rho - \tfrac{1}{2}\ell\pi),\; n_\ell(\rho) \sim -\frac{1}{\rho}\cos(\rho - \tfrac{1}{2}\ell\pi) \quad \text{as } \rho \to \infty\,,\\
j_\ell(\rho) &\propto \rho^\ell,\; n_\ell(\rho) \propto \rho^{-\ell-1} \quad \text{as } \rho \to 0\,,\\
e^{i\rho\cos\theta} &= \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell\, j_\ell(\rho)\, P_\ell(\cos\theta)\,,\\
j_0(\rho) &= \frac{\sin\rho}{\rho}, \qquad n_0(\rho) = -\frac{\cos\rho}{\rho}\,.
\end{aligned}$$

]

24C General Relativity

(i) State and prove Birkhoff's theorem.

(ii) Derive the Schwarzschild metric and discuss its relevance to the problem of gravitational collapse and the formation of black holes.

[*Hint: You may assume that the metric takes the form*

$$ds^2 = -e^{\nu(r,t)}\,dt^2 + e^{\lambda(r,t)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

and that the non-vanishing components of the Einstein tensor are given by

$$G_{tt} = \frac{e^{2\nu+\lambda}}{r^2}(-1 + e^{\lambda} + r\lambda'), \quad G_{rt} = e^{(\nu+\lambda)/2}\,\frac{\dot\lambda}{r}, \quad G_{rr} = \frac{e^{\lambda}}{r^2}(1 - e^{-\lambda} + r\nu'),$$

$$G_{\theta\theta} = \tfrac{1}{4}r^2e^{-\lambda}\Big[2\nu'' + (\nu')^2 + \frac{2}{r}(\nu' - \lambda') - \nu'\lambda'\Big] - \tfrac{1}{4}r^2e^{-\nu}\big[2\ddot\lambda + (\dot\lambda)^2 - \dot\lambda\dot\nu\big],$$

$G_{tr} = G_{rt}$ *and* $G_{\phi\phi} = \sin^2\theta\,G_{\theta\theta}$.]

25A Fluid Dynamics II

An incompressible fluid with density ρ and viscosity μ is forced by a pressure difference Δp through the narrow gap between two parallel circular cylinders of radius a with axes $2a + b$ apart. Explaining any approximations made, show that, provided $b \ll a$ and $\rho b^3 \Delta p \ll \mu^2 a$, the volume flux (per unit length of cylinder) is

$$\frac{2b^{5/2}\Delta p}{9\pi a^{1/2}\mu}$$

when the cylinders are stationary.

Show also that when the two cylinders rotate with angular velocities Ω and $-\Omega$ respectively, the change in the volume flux is

$$\frac{4}{3}ba\Omega.$$

For the case $\Delta p = 0$, find and sketch the function $f(x) = u_0(x)/(a\Omega)$, where u_0 is the centreline velocity at position x along the gap in the direction of flow. Comment on the values taken by f.

26A Waves in Fluid and Solid Media

The linearised equation of motion governing small disturbances in a homogeneous elastic medium of density ρ is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}\,,$$

where $\mathbf{u}(\mathbf{x}, t)$ is the displacement, and λ and μ are the Lamé constants. Derive solutions for plane longitudinal waves P with wavespeed c_P, and plane shear waves S with wavespeed c_S.

The half-space $y < 0$ is filled with the elastic solid described above, while the slab $0 < y < h$ is filled with an elastic solid with shear modulus $\overline{\mu}$, and wavespeeds $\overline{c}_P$ and $\overline{c}_S$. There is a vacuum in $y > h$. A harmonic plane SH wave of frequency ω and unit amplitude propagates from $y < 0$ towards the interface $y = 0$. The wavevector is in the xy-plane, and makes an angle θ with the y-axis. Derive the complex amplitude, R, of the reflected SH wave in $y < 0$. Evaluate $|R|$ for all possible values of $\overline{c}_S/c_S$, and explain your answer.