UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Monday 31 May 2004 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

The number of marks for each question is the same.

Additional credit will be given for a substantially complete answer.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **3G, 6G** *should be in one bundle and* **1I, 13I** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1I Markov Chains

(i) Give the definitions of a *recurrent* and a *null recurrent* irreducible Markov chain.

Let (X_n) be a recurrent Markov chain with state space I and irreducible transition matrix $P = (p_{ij})$. Prove that the vectors $\gamma^k = (\gamma_j^k,\ j \in I)$, $k \in I$, with entries $\gamma_k^k = 1$ and

$$\gamma_i^k = \mathbb{E}_k(\#\text{ of visits to } i \text{ before returning to } k), \quad i \neq k\,,$$

are P-invariant:

$$\gamma_j^k = \sum_{i \in I} \gamma_i^k p_{ij}\,.$$

(ii) Let (W_n) be the birth and death process on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with the following transition probabilities:

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{2},\ i \geq 1$$
$$p_{01} = 1\,.$$

By relating (W_n) to the symmetric simple random walk (Y_n) on $\mathbb{Z}$, or otherwise, prove that (W_n) is a recurrent Markov chain. By considering invariant measures, or otherwise, prove that (W_n) is null recurrent.

Calculate the vectors $\gamma^k = (\gamma_i^k, i \in \mathbb{Z}_+)$ for the chain (W_n), $k \in \mathbb{Z}_+$.

Finally, let $W_0 = 0$ and let N be the number of visits to 1 before returning to 0. Show that $\mathbb{P}_0(N = n) = (1/2)^n$, $n \geq 1$.

[*You may use properties of the random walk* (Y_n) *or general facts about Markov chains without proof but should clearly state them.*]

2B Principles of Dynamics

(i) In Hamiltonian mechanics the action is written

$$S = \int dt \Big(p^a \dot{q}^a - H(q^a, p^a, t) \Big) \,. \qquad (1)$$

Starting from Maupertius' principle $\delta S = 0$, derive Hamilton's equations

$$\dot{q}^a = \frac{\partial H}{\partial p^a}, \quad \dot{p}^a = -\frac{\partial H}{\partial q^a} \,.$$

Show that H is a constant of the motion if $\partial H/\partial t = 0$. When is p^a a constant of the motion?

(ii) Consider the action S given in Part (i), evaluated on a classical path, as a function of the final coordinates q^a_f and final time t_f, with the initial coordinates and the initial time held fixed. Show that $S(q^a_f, t_f)$ obeys

$$\frac{\partial S}{\partial q^a_f} = p^a_f, \quad \frac{\partial S}{\partial t_f} = -H(q^a_f, p^a_f, t_f) \,. \qquad (2)$$

Now consider a simple harmonic oscillator with $H = \frac{1}{2}(p^2 + q^2)$. Setting the initial time and the initial coordinate to zero, find the classical solution for p and q with final coordinate $q = q_f$ at time $t = t_f$. Hence calculate $S(t_f, q_f)$, and explicitly verify (2) in this case.

3G Groups, Rings and Fields

(i) Let R be a commutative ring. Define the terms *prime ideal* and *maximal ideal*, and show that if an ideal M in R is maximal then M is also prime.

(ii) Let P be a non-trivial prime ideal in the commutative ring R ('non-trivial' meaning that $P \neq \{0\}$ and $P \neq R$). If P has finite index as a subgroup of R, show that P is also maximal. Give an example to show that this may fail, if the assumption of finite index is omitted. Finally, show that if R is a principal ideal domain, then every non-trivial prime ideal in R is maximal.

4C Electromagnetism

(i) Show that the work done in assembling a localised charge distribution $\rho(\mathbf{r})$ in a region V with an associated potential $\phi(\mathbf{r})$ is

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\phi(\mathbf{r})\,d\tau,$$

and that this can be written as an integral over all space

$$W = \frac{1}{2}\epsilon_0 \int |\mathbf{E}|^2\,d\tau,$$

where the electric field $\mathbf{E} = -\nabla\phi$.

(ii) What is the force per unit area on an infinite plane conducting sheet with a charge density σ per unit area (a) if it is isolated in space and (b) if the electric field vanishes on one side of the sheet?

An infinite cylindrical capacitor consists of two concentric cylindrical conductors with radii a, b $(a < b)$, carrying charges $\pm q$ per unit length respectively. Calculate the capacitance per unit length and the energy per unit length. Next determine the total force on each conductor, and calculate the rate of change of energy of the inner and outer conductors if they are moved radially inwards and outwards respectively with speed v. What is the corresponding rate of change of the capacitance?

5F Combinatorics

State and prove Menger's theorem (vertex form).

Let G be a graph of connectivity $\kappa(G) \geq k$ and let S, T be disjoint subsets of $V(G)$ with $|S|, |T| \geq k$. Show that there exist k vertex disjoint paths from S to T.

The graph H is said to be k-*linked* if, for every sequence $s_1, \ldots, s_k, t_1, \ldots, t_k$ of $2k$ distinct vertices, there exist $s_i - t_i$ paths, $1 \leq i \leq k$, that are vertex disjoint. By removing an edge from K_{2k}, or otherwise, show that, for $k \geqslant 2$, H need not be k-linked even if $\kappa(H) \geq 2k - 2$.

Prove that if $|H| = n$ and $\delta(H) \geq \frac{1}{2}(n + 3k) - 2$ then H is k-linked.

6G Representation Theory

(a) Show that every irreducible complex representation of an abelian group is one-dimensional.

(b) Show, by example, that the analogue of (a) fails for real representations.

(c) Let the cyclic group of order n act on $\mathbb{C}^n$ by cyclic permutation of the standard basis vectors. Decompose this representation explicitly into irreducibles.

7H Galois Theory

Let L/K be a finite extension of fields. Define the *trace* $\mathrm{Tr}_{L/K}(x)$ and *norm* $N_{L/K}(x)$ of an element $x \in L$.

Assume now that the extension L/K is Galois, with cyclic Galois group of prime order p, generated by σ.

i) Show that $\mathrm{Tr}_{L/K}(x) = \sum_{n=0}^{p-1} \sigma^n(x)$.

ii) Show that $\{\sigma(x) - x \mid x \in L\}$ is a K-vector subspace of L of dimension $p-1$. Deduce that if $y \in L$, then $\mathrm{Tr}_{L/K}(y) = 0$ if and only if $y = \sigma(x) - x$ for some $x \in L$. [*You may assume without proof that* $\mathrm{Tr}_{L/K}$ *is surjective for any finite separable extension* L/K.]

iii) Suppose that L has characteristic p. Deduce from (i) that every element of K can be written as $\sigma(x) - x$ for some $x \in L$. Show also that if $\sigma(x) = x + 1$, then $x^p - x$ belongs to K. Deduce that L is the splitting field over K of $X^p - X - a$ for some $a \in K$.

8H Differentiable Manifolds

What is a *smooth vector bundle* over a manifold M?

Assuming the existence of "bump functions", prove that every compact manifold embeds in some Euclidean space $\mathbb{R}^n$.

By choosing an inner product on $\mathbb{R}^n$, or otherwise, deduce that for any compact manifold M there exists some vector bundle $\eta \to M$ such that the direct sum $TM \oplus \eta$ is isomorphic to a trivial vector bundle.

9H Number Fields

Let $K = \mathbb{Q}(\theta)$, where θ is a root of $X^3 - 4X + 1$. Prove that K has degree 3 over $\mathbb{Q}$, and admits three distinct embeddings in $\mathbb{R}$. Find the discriminant of K and determine the ring of integers $\mathcal{O}$ of K. Factorise $2\mathcal{O}$ and $3\mathcal{O}$ into a product of prime ideals.

Using Minkowski's bound, show that K has class number 1 provided all prime ideals in $\mathcal{O}$ dividing 2 and 3 are principal. Hence prove that K has class number 1.

[*You may assume that the discriminant of* $X^3 + aX + b$ *is* $-4a^3 - 27b^2$.]

10G Hilbert Spaces

Suppose that (e_n) and (f_m) are orthonormal bases of a Hilbert space H and that $T \in L(H)$.

(a) Show that $\sum_{n=1}^{\infty} \|T(e_n)\|^2 = \sum_{m=1}^{\infty} \|T^*(f_m)\|^2$.

(b) Show that $\sum_{n=1}^{\infty} \|T(e_n)\|^2 = \sum_{m=1}^{\infty} \|T(f_m)\|^2$.

$T \in L(H)$ is a *Hilbert-Schmidt* operator if $\sum_{n=1}^{\infty} \|T(e_n)\|^2 < \infty$ for some (and hence every) orthonormal basis (e_n).

(c) Show that the set HS of Hilbert-Schmidt operators forms a linear subspace of $L(H)$, and that $\langle T, S\rangle = \sum_{n=1}^{\infty} \langle T(e_n), S(e_n)\rangle$ is an inner product on *HS*; show that this inner product does not depend on the choice of the orthonormal basis (e_n).

(d) Let $\|T\|_{HS}$ be the corresponding norm. Show that $\|T\| \leqslant \|T\|_{HS}$, and show that a Hilbert-Schmidt operator is compact.

11H Riemann Surfaces

Let τ be a fixed complex number with positive imaginary part. For $z \in \mathbb{C}$, define

$$v(z) = \sum_{n=-\infty}^{\infty} \exp\bigl(\pi i \tau n^2 + 2\pi i n(z + \tfrac{1}{2})\bigr).$$

Prove the identities

$$v(z+1) = v(z), \qquad v(-z) = v(z), \qquad v(z+\tau) = -\exp(-\pi i \tau - 2\pi i z) \cdot v(z)$$

and deduce that $v(\tau/2) = 0$. Show further that $\tau/2$ is the *only* zero of v in the parallelogram P with vertices $-1/2$, $1/2$, $1/2+\tau$, $-1/2+\tau$.

[*You may assume that v is holomorphic on* $\mathbb{C}$.]

Now let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ be two sets of complex numbers and

$$f(z) = \prod_{j=1}^{k} \frac{v(z - a_j)}{v(z - b_j)}.$$

Prove that f is a doubly-periodic meromorphic function, with periods 1 and τ, if and only if $\sum_{j=1}^{k} (a_j - b_j)$ is an integer.

12F Logic, Computation and Set Theory

(i) State and prove the Knaster-Tarski Fixed-Point Theorem.

(ii) A subset S of a poset X is called an *up-set* if whenever $x, y \in X$ satisfy $x \in S$ and $x \leqslant y$ then also $y \in S$. Show that the set of up-sets of X (ordered by inclusion) is a complete poset.

Let X and Y be totally ordered sets, such that X is isomorphic to an up-set in Y and Y is isomorphic to the complement of an up-set in X. Prove that X is isomorphic to Y. Indicate clearly where in your argument you have made use of the fact that X and Y are total orders, rather than just partial orders.

[Recall that posets X and Y are called *isomorphic* if there exists a bijection f from X to Y such that, for any $x, y \in X$, we have $f(x) \leqslant f(y)$ if and only if $x \leqslant y$.]

13I Probability and Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{A} = (A_i : i = 1, 2, \ldots)$ be a sequence of events.

(a) What is meant by saying that $\mathcal{A}$ is a family of *independent* events?

(b) Define the events $\{A_n \text{ infinitely often}\}$ and $\{A_n \text{ eventually}\}$. State and prove the two Borel–Cantelli lemmas for $\mathcal{A}$.

(c) Let $X_1, X_2, \ldots$ be the outcomes of a sequence of independent flips of a fair coin,

$$\mathbb{P}(X_i = 0) \;=\; \mathbb{P}(X_i = 1) = \tfrac{1}{2} \qquad \text{for } i \geqslant 1.$$

Let L_n be the length of the run beginning at the n^{th} flip. For example, if the first fourteen outcomes are 01110010000110, then $L_1 = 1$, $L_2 = 3$, $L_3 = 2$, etc.

Show that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{L_n}{\log_2 n} > 1\right) = 0\ ,$$

and furthermore that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{L_n}{\log_2 n} = 1\right) = 1\ .$$

14J Information Theory

State the formula for the capacity of a memoryless channel.

(a) Consider a memoryless channel where there are two input symbols, A and B, and three output symbols, $A, B, *$. Suppose each input symbol is left intact with probability $1/2$, and transformed into a $*$ with probability $1/2$. Write down the channel matrix, and calculate the capacity.

(b) Now suppose the output is further processed by someone who cannot distinguish A and $*$, so that the matrix becomes

$$\begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} .$$

Calculate the new capacity.

15J Principles of Statistics

(i) What does it mean to say that a family $\{f(\cdot|\theta) : \theta \in \Theta\}$ of densities is an *exponential family?*

Consider the family of densities on $(0, \infty)$ parametrised by the positive parameters a, b and defined by

$$f(x|a, b) = \frac{a \exp(-(a - bx)^2/2x)}{\sqrt{2\pi x^3}} \qquad (x > 0).$$

Prove that this family is an exponential family, and identify the natural parameters and the reference measure.

(ii) Let $(X_1, \ldots, X_n)$ be a sample drawn from the above distribution. Find the maximum-likelihood estimators of the parameters (a, b). Find the Fisher information matrix of the family (in terms of the natural parameters). Briefly explain the significance of the Fisher information matrix in relation to unbiased estimation. Compute the mean of X_1 and of X_1^{-1}.

16J Stochastic Financial Models

(i) What does it mean to say that U is a *utility function*? What is a utility function with constant absolute risk aversion (CARA)?

Let $S_t \equiv (S_t^1, \ldots, S_t^d)^T$ denote the prices at time $t = 0, 1$ of d risky assets, and suppose that there is also a riskless zeroth asset, whose price at time 0 is 1, and whose price at time 1 is $1+r$. Suppose that S_1 has a multivariate Gaussian distribution, with mean μ_1 and non-singular covariance V. An agent chooses at time 0 a portfolio $\theta = (\theta^1, \ldots, \theta^d)^T$ of holdings of the d risky assets, at total cost $\theta \cdot S_0$, and at time 1 realises his gain $X = \theta \cdot (\, S_1 - (1+r)S_0 \,)$. Given that he wishes the mean of X to be equal to m, find the smallest value that the variance v of X can be. What is the portfolio that achieves this smallest variance? Hence sketch the region in the (v, m) plane of pairs (v, m) that can be achieved by some choice of θ, and indicate the mean-variance efficient frontier.

(ii) Suppose that the agent has a CARA utility with coefficient γ of absolute risk aversion. What portfolio will he choose in order to maximise $EU(X)$? What then is the mean of X?

Regulation requires that the agent's choice of portfolio θ has to satisfy the value-at-risk (VaR) constraint

$$m \geqslant -L + a\sqrt{v},$$

where $L > 0$ and $a > 0$ are determined by the regulatory authority. Show that this constraint has no effect on the agent's decision if $\kappa \equiv \sqrt{\mu \cdot V^{-1}\mu} \geqslant a$. If $\kappa < a$, will this constraint necessarily affect the agent's choice of portfolio?

17B Nonlinear Dynamical Systems

(i) State Liapunov's First Theorem and La Salle's Invariance Principle. Use these results to show that the system

$$\ddot{x} + k\dot{x} + \sin x = 0, \quad k > 0$$

has an asymptotically stable fixed point at the origin.

(ii) Define the *basin of attraction* of an invariant set of a dynamical system.

Consider the equations

$$\dot{x} = -x + \beta x y^2 + x^3, \quad \dot{y} = -y + \beta y x^2 + y^3, \quad \beta > 2.$$

(a) Find the fixed points of the system and determine their type.

(b) Show that the basin of attraction of the origin includes the union over α of the regions

$$x^2 + \alpha^2 y^2 < \frac{4\alpha^2(1+\alpha^2)(\beta-1)}{\beta^2(1+\alpha^2)^2 - 4\alpha^2}.$$

Sketch these regions for $\alpha^2 = 1, 1/2, 2$ in the case $\beta = 3$.

18D Partial Differential Equations

(a) State and prove the Mean Value Theorem for harmonic functions.

(b) Let $u \geqslant 0$ be a harmonic function on an open set $\Omega \subset \mathbb{R}^n$. Let $B(x,a) = \{y \in \mathbb{R}^n : |x-y| < a\}$. For any $x \in \Omega$ and for any $r > 0$ such that $B(x,4r) \subset \Omega$, show that

$$\sup_{\{y \in B(x,r)\}} u(y) \leqslant 3^n \inf_{\{y \in B(x,r)\}} u(y) .$$

19D Methods of Mathematical Physics

State the convolution theorem for Laplace transforms.

The temperature $T(x,t)$ in a semi-infinite rod satisfies the heat equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t}, \quad x \geq 0, \; t \geq 0$$

and the conditions $T(x,0) = 0$ for $x \geq 0$, $T(0,t) = f(t)$ for $t \geq 0$ and $T(x,t) \to 0$ as $x \to \infty$. Show that

$$T(x,t) = \int_0^t f(\tau)\, G(x,t-\tau) d\tau,$$

where

$$G(x,t) = \sqrt{\frac{x^2}{4\pi k t^3}}\, e^{-x^2/4kt} \; .$$

20D Numerical Analysis

(i) Define the Backward Difference Formula (BDF) method for ordinary differential equations and derive its two-step version.

(ii) Prove that the interval $(-\infty, 0)$ belongs to the linear stability domain of the two-step BDF method.

21C Electrodynamics

The Maxwell field tensor is

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$

and the 4-current density is $J^a = (\rho, \mathbf{j})$. Write down the 3-vector form of Maxwell's equations and the continuity equation, and obtain the equivalent 4-vector equations.

Consider a Lorentz transformation from a frame $\mathcal{F}$ to a frame $\mathcal{F}'$ moving with relative (coordinate) velocity v in the x-direction

$$L^a{}_b = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\gamma = 1/\sqrt{1-v^2}$. Obtain the transformation laws for $\mathbf{E}$ and $\mathbf{B}$. Which quantities, quadratic in $\mathbf{E}$ and $\mathbf{B}$, are Lorentz scalars?

22E Statistical Physics

Define the notions of entropy S and thermodynamic temperature T for a gas of particles in a variable volume V. Derive the fundamental relation

$$dE \;=\; TdS - PdV\;.$$

The free energy of the gas is defined as $F = E - TS$. Why is it convenient to regard F as a function of T and V? By considering F, or otherwise, show that

$$\left.\frac{\partial S}{\partial V}\right|_T \;=\; \left.\frac{\partial P}{\partial T}\right|_V\;.$$

Deduce that the entropy of an ideal gas, whose equation of state is $PV = NT$ (using energy units), has the form

$$S \;=\; N\log\left(\frac{V}{N}\right) \;+\; Nc(T)\;,$$

where $c(T)$ is independent of N and V.

Show that if the gas is in contact with a heat bath at temperature T, then the probability of finding the gas in a particular quantum microstate of energy E_r is

$$P_r \;=\; e^{(F-E_r)/T}\;.$$

23E Applications of Quantum Mechanics

The operator corresponding to a rotation through an angle θ about an axis $\mathbf{n}$, where $\mathbf{n}$ is a unit vector, is

$$U(\mathbf{n},\theta) = e^{i\theta\, \mathbf{n}\cdot\mathbf{J}/\hbar}\,.$$

If U is unitary show that $\mathbf{J}$ must be hermitian. Let $\mathbf{V} = (V_1, V_2, V_3)$ be a vector operator such that

$$U(\mathbf{n},\delta\theta)\mathbf{V}U(\mathbf{n},\delta\theta)^{-1} = \mathbf{V} + \delta\theta\, \mathbf{n}\times\mathbf{V}\,.$$

Work out the commutators $[J_i, V_j]$. Calculate

$$U(\hat{\mathbf{z}},\theta)\mathbf{V}U(\hat{\mathbf{z}},\theta)^{-1}\,,$$

for each component of $\mathbf{V}$.

If $|jm\rangle$ are standard angular momentum states determine $\langle jm'|U(\hat{\mathbf{z}},\theta)|jm\rangle$ for any j, m, m' and also determine $\langle \frac{1}{2}m'|U(\hat{\mathbf{y}},\theta)|\frac{1}{2}m\rangle$.

$\big[$*Hint* : $J_3|jm\rangle = m\hbar|jm\rangle,\ J_+|\frac{1}{2}\,{-\frac{1}{2}}\rangle = \hbar|\frac{1}{2}\,\frac{1}{2}\rangle,\ J_-|\frac{1}{2}\,\frac{1}{2}\rangle = \hbar|\frac{1}{2}\,{-\frac{1}{2}}\rangle.\big]$

24C General Relativity

(i) What is an affine parameter λ of a timelike or null geodesic? Prove that for a timelike geodesic one may take λ to be proper time τ. The metric

$$ds^2 = -dt^2 + a^2(t)\, d\mathbf{x}^2,$$

with $\dot{a}(t) > 0$ represents an expanding universe. Calculate the Christoffel symbols.

(ii) Obtain the law of spatial momentum conservation for a particle of rest mass m in the form

$$ma^2\,\frac{d\mathbf{x}}{d\tau} = \mathbf{p} = \text{constant}.$$

Assuming that the energy $E = m\, dt/d\tau$, derive an expression for E in terms of m, $\mathbf{p}$ and $a(t)$ and show that the energy is not conserved but rather that it decreases with time. In particular, show that if the particle is moving extremely relativistically then the energy decreases as $a^{-1}(t)$, and if it is moving non-relativistically then the kinetic energy, $E - m$, decreases as $a^{-2}(t)$.

Show that the frequency ω_e of a photon emitted at time t_e will be observed at time t_o to have frequency

$$\omega_o = \omega_e\,\frac{a(t_e)}{a(t_o)}.$$

25A Fluid Dynamics II

Consider a uniform stream of inviscid incompressible fluid incident onto a two-dimensional body (such as a circular cylinder). Sketch the flow in the region close to the stagnation point, S, at the front of the body.

Let the fluid now have a small but non-zero viscosity. Using local co-ordinates x along the boundary and y normal to it, with the stagnation point as origin and $y > 0$ in the fluid, explain why the local outer, inviscid flow is approximately of the form

$$\mathbf{u} = (Ex, -Ey)$$

for some positive constant E.

Use scaling arguments to find the thickness δ of the boundary layer on the body near S. Hence show that there is a solution of the boundary layer equations of the form

$$u(x, y) = Exf'(\eta),$$

where η is a suitable similarity variable and f satisfies

$$f''' + ff'' - f'^2 = -1. \qquad (*)$$

What are the appropriate boundary conditions for $(*)$ and why? Explain *briefly* how you would obtain a numerical solution to $(*)$ subject to the appropriate boundary conditions.

Explain why it is neither possible nor appropriate to perform a similar analysis near the rear stagnation point of the inviscid flow.

26A Waves in Fluid and Solid Media

A physical system permits one-dimensional wave propagation in the x-direction according to the equation

$$\frac{\partial^2 \psi}{\partial t^2} - \alpha^2 \frac{\partial^6 \psi}{\partial x^6} = 0\,,$$

where α is a real positive constant. Derive the corresponding dispersion relation and sketch graphs of frequency, phase velocity and group velocity as functions of the wave number. Is it the shortest or the longest waves that are at the front of a dispersing wave train arising from a localised initial disturbance? Do the wave crests move faster or slower than a packet of waves?

Find the solution of the above equation for the initial disturbance given by

$$\psi(x, 0) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk\,, \qquad \frac{\partial \psi}{\partial t}(x, 0) = 0\,,$$

where $A(k)$ is real and $A(-k) = A(k)$.

Use the method of stationary phase to obtain a leading-order approximation to this solution for large t when $V = x/t$ is held fixed.

[*Note that*

$$\int_{-\infty}^{\infty} e^{\pm iu^2} du = \pi^{\frac{1}{2}} e^{\pm i\pi/4}\,. \qquad]$$