UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Thursday 3 June 2004 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)***, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)***.*

Additional credit will be given for a substantially complete answer to either Part.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **5C, 14C** *should be in one bundle and* **10J, 12J** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1I Markov Chains

(i) Give the definition of the *time-reversal* of a discrete-time Markov chain (X_n). Define a *reversible* Markov chain and check that every probability distribution satisfying the detailed balance equations is invariant.

(ii) Customers arrive in a hairdresser's shop according to a Poisson process of rate $\lambda > 0$. The shop has s hairstylists and N waiting places; each stylist is working (on a single customer) provided that there is a customer to serve, and any customer arriving when the shop is full (i.e. the numbers of customers present is $N + s$) is not admitted and never returns. Every admitted customer waits in the queue and then is served, in the first-come-first-served order (say), the service taking an exponential time of rate $\mu > 0$; the service times of admitted customers are independent. After completing his/her service, the customer leaves the shop and never returns.

Set up a Markov chain model for the number X_t of customers in the shop at time $t \geq 0$. Assuming $\lambda < s\mu$, calculate the equilibrium distribution π of this chain and explain why it is unique. Show that (X_t) in equilibrium is time-reversible, i.e. $\forall\, T > 0, (X_t, 0 \leq t \leq T)$ has the same distribution as $(Y_t, 0 \leq t \leq T)$ where $Y_t = X_{T-t}$, and $X_0 \sim \pi$.

2B Principles of Dynamics

(i) Explain the concept of a *canonical transformation* from coordinates (q^a, p^a) to (Q^a, P^a). Derive the transformations corresponding to generating functions $F_1(t, q^a, Q^a)$ and $F_2(t, q^a, P^a)$.

(ii) A particle moving in an electromagnetic field is described by the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - e\Big(\phi - \frac{\dot{\mathbf{x}} \cdot \mathbf{A}}{c}\Big),$$

where c is constant.

(a) Derive the equations of motion in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

(b) Show that $\mathbf{E}$ and $\mathbf{B}$ are invariant under the gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda, \quad \phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \tag{1}$$

for arbitrary $\Lambda(t, \mathbf{x})$.

(c) Construct the Hamiltonian. Find the generating function F_2 for the canonical transformation which implements the gauge transformation (1).

3F Functional Analysis

(i) Let H be an infinite-dimensional Hilbert space. Show that H has a (countable) orthonormal basis if and only if H has a countable dense subset. [*You may assume familiarity with the Gram-Schmidt process.*]

State and prove Bessel's inequality.

(ii) State Parseval's equation. Using this, prove that if H has a countable dense subset then there is a surjective isometry from H to l^2.

Explain carefully why the functions $e^{in\theta}$, $n \in \mathbb{Z}$, form an orthonormal basis for $L^2(\mathbb{T})$.

4G Groups, Rings and Fields

(i) Let $K \leqslant \mathbb{C}$ be a field and $L \leqslant \mathbb{C}$ a finite normal extension of K. If H is a finite subgroup of order m in the Galois group $G(L \mid K)$, show that L is a normal extension of the H-invariant subfield $I(H)$ of degree m and that $G(L \mid I(H)) = H$. [*You may assume the theorem of the primitive element.*]

(ii) Show that the splitting field over $\mathbb{Q}$ of the polynomial x^4+2 is $\mathbb{Q}[\sqrt[4]{2}, i]$ and deduce that its Galois group has order 8. Exhibit a subgroup of order 4 of the Galois group, and determine the corresponding invariant subfield.

5C Electromagnetism

(i) State Maxwell's equations and show that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ can be expressed in terms of a scalar potential ϕ and a vector potential $\mathbf{A}$. Hence derive the inhomogeneous wave equations that are satisfied by ϕ and $\mathbf{A}$ respectively.

(ii) The plane $x = 0$ separates a vacuum in the half-space $x < 0$ from a perfectly conducting medium occupying the half-space $x > 0$. Derive the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at $x = 0$.

A plane electromagnetic wave with a magnetic field $\mathbf{B} = B(t, x, z)\hat{\mathbf{y}}$, travelling in the xz-plane at an angle θ to the x-direction, is incident on the interface at $x = 0$. If the wave has frequency ω show that the total magnetic field is given by

$$\mathbf{B} = B_0 \cos\left(\tfrac{\omega x}{c}\cos\theta\right) \exp\left[i\left(\tfrac{\omega z}{c}\sin\theta - \omega t\right)\right]\hat{\mathbf{y}},$$

where B_0 is a constant. Hence find the corresponding electric field $\mathbf{E}$, and obtain the surface charge density and the surface current at the interface.

6B Nonlinear Dynamical Systems

(i) Consider a system in $\mathbb{R}^2$ that is *almost Hamiltonian*:

$$\dot{x} = \frac{\partial H}{\partial y} + \epsilon g_1(x, y), \quad \dot{y} = -\frac{\partial H}{\partial x} + \epsilon g_2(x, y) \ ,$$

where $H = H(x, y)$ and $|\epsilon| \ll 1$. Show that if the system has a periodic orbit $\mathcal{C}$ then $\oint_{\mathcal{C}} g_2 dx - g_1 dy = 0$, and explain how to evaluate this orbit approximately for small ϵ. Illustrate your method by means of the system

$$\dot{x} = y + \epsilon x(1 - x^2), \quad \dot{y} = -x.$$

(ii) Consider the system

$$\dot{x} = y, \quad \dot{y} = x - x^3 + \epsilon y(1 - \alpha x^2).$$

(a) Show that when $\epsilon = 0$ the system is Hamiltonian, and find the Hamiltonian. Sketch the trajectories in the case $\epsilon = 0$. Identify the value H_c of H for which there is a homoclinic orbit.

(b) Suppose $\epsilon > 0$. Show that the small change ΔH in H around an orbit of the Hamiltonian system can be expressed to leading order as an integral of the form

$$\int_{x_1}^{x_2} \mathcal{F}(x, H) dx,$$

where x_1, x_2 are the extrema of the x-coordinates of the orbits of the Hamiltonian system, distinguishing between the cases $H < H_c$, $H > H_c$.

(c) Find the value of α, correct to leading order in ϵ, at which the system has a homoclinic orbit.

(d) By examining the eigenvalues of the Jacobian at the origin, determine the stability of the homoclinic orbit, being careful to state clearly any standard results that you use.

7G Geometry of Surfaces

(i) The *catenoid* is the surface C in Euclidean $\mathbb{R}^3$, with co-ordinates x, y, z and Riemannian metric $ds^2 = dx^2 + dy^2 + dz^2$ obtained by rotating the curve $y = \cosh x$ about the x-axis, while the *helicoid* is the surface H swept out by a line which lies along the x-axis at time $t = 0$, and at time $t = t_0$ is perpendicular to the z-axis, passes through the point $(0, 0, t_0)$ and makes an angle t_0 with the x-axis.

Find co-ordinates on each of C and H and write x, y, z in terms of these co-ordinates.

(ii) Compute the induced Riemannian metrics on C and H in terms of suitable co-ordinates. Show that H and C are locally isometric. By considering the x-axis in H, show that this local isometry cannot be extended to a rigid motion of any open subset of Euclidean $\mathbb{R}^3$.

8F Logic, Computation and Set Theory

(i) State and prove the Compactness Theorem for first-order predicate logic.

State and prove the Upward Löwenheim-Skolem Theorem.

[*You may use the Completeness Theorem for first-order predicate logic.*]

(ii) For each of the following theories, either give axioms (in the language of posets) for the theory or prove carefully that the theory is not axiomatisable.

(a) The theory of posets having no maximal element.

(b) The theory of posets having a unique maximal element.

(c) The theory of posets having infinitely many maximal elements.

(d) The theory of posets having finitely many maximal elements.

(e) The theory of countable posets having a unique maximal element.

9H Number Theory

(i) Find a solution in integers of the Pell equation $x^2 - 17y^2 = 1$.

(ii) Define the continued fraction expansion of a real number $\theta > 1$ and show that it converges to θ.

Show that if $N > 0$ is a nonsquare integer and x and y are integer solutions of $x^2 - Ny^2 = 1$, then x/y is a convergent of $\sqrt{N}$.

10J Algorithms and Networks

(i) Consider the problem

$$\begin{array}{rl} \text{minimise} & f(x) \\ \text{subject to} & g(x) = b,\ x \in X, \end{array} \qquad (*)$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R},\ g : \mathbb{R}^n \longrightarrow \mathbb{R}^m,\ X \subseteq \mathbb{R}^n,\ x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. State the Lagrange Sufficiency Theorem for problem $(*)$. What is meant by saying that this problem is *strong Lagrangian*? How is this related to the Lagrange Sufficiency Theorem? Define a *supporting hyperplane* and state a condition guaranteeing that problem $(*)$ is strong Lagrangian.

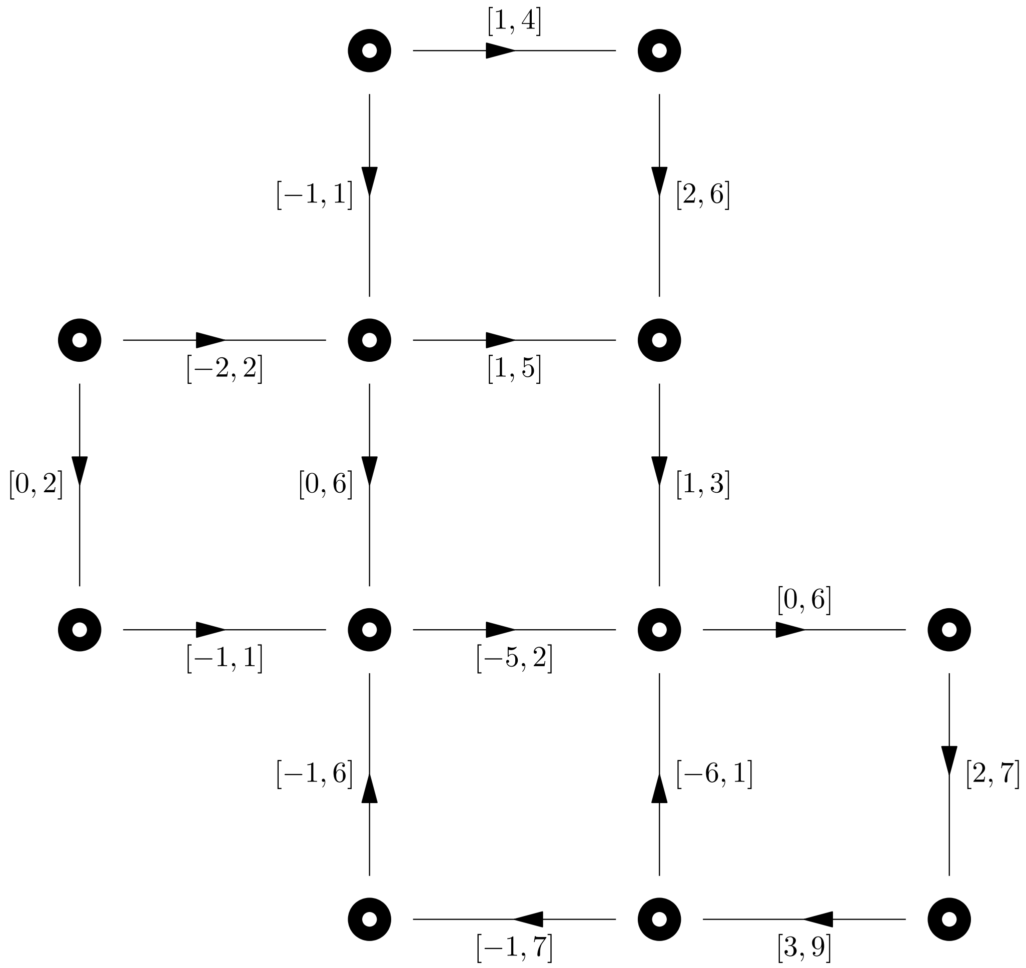
(ii) Define the terms *flow, divergence, circulation, potential* and *differential* for a network with nodes N and arcs A.

State the feasible differential problem for a network with span intervals $D(j) = [d^-(j), d^+(j)],\ j \in A$.

State, without proof, the Feasible Differential Theorem.

[*You must carefully define all quantities used in your statements.*]

Show that the network below does not support a feasible differential.



11J Stochastic Financial Models

(i) Consider a single-period binomial model of a riskless asset (asset 0), worth 1 at time 0 and $1+r$ at time 1, and a risky asset (asset 1), worth 1 at time 0 and worth u at time 1 if the period was good, otherwise worth d. Assuming that

$$d < 1 + r < u \qquad (*)$$

show how any contingent claim Y to be paid at time 1 can be priced and exactly replicated. Briefly explain the significance of the condition $(*)$, and indicate how the analysis of the single-period model extends to many periods.

(ii) Now suppose that $u = 5/3$, $d = 2/3$, $r = 1/3$, and that the risky asset is worth $S_0 = 864 = 2^5 \times 3^3$ at time zero. Show that the time-0 value of an American put option with strike $K = S_0$ and expiry at time $t = 3$ is equal to 79, and find the optimal exercise policy.

12J Principles of Statistics

(i) What is a *sufficient statistic*? What is a *minimal* sufficient statistic? Explain the terms *nuisance parameter* and *ancillary statistic*.

(ii) Let $U_1, \ldots, U_n$ be independent random variables with common uniform$([0,1])$ distribution, and suppose you observe $X_i \equiv aU_i^{-\beta}$, $i = 1, \ldots, n$, where the positive parameters a, β are unknown. Write down the joint density of $X_1, \ldots, X_n$ and prove that the statistic

$$(m, p) \equiv \Big(\min_{1 \leqslant j \leqslant n}\{X_j\}, \prod_{j=1}^{n} X_j\Big)$$

is minimal sufficient for (a, β). Find the maximum-likelihood estimator $(\hat{a}, \hat{\beta})$ of (a, β).

Regarding β as the parameter of interest and a as the nuisance parameter, is m ancillary? Find the mean and variance of $\hat{\beta}$. Hence find an unbiased estimator of β.

13E Foundations of Quantum Mechanics

(i) A quantum mechanical system consists of two identical non-interacting particles with associated single-particle wave functions $\psi_i(x)$ and energies E_i, $i = 1, 2, \ldots$, where $E_1 < E_2 < \ldots$. Show how the states for the two lowest energy levels of the system are constructed and discuss their degeneracy when the particles have (a) spin 0, (b) spin 1/2.

(ii) The Pauli matrices are defined to be

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

State how the spin operators s_1, s_2, s_3 may be expressed in terms of the Pauli matrices, and show that they describe states with total angular momentum $\frac{1}{2}\hbar$.

An electron is at rest in the presence of a magnetic field $\mathbf{B} = (B, 0, 0)$, and experiences an interaction potential $-\mu\boldsymbol{\sigma} \cdot \mathbf{B}$. At $t = 0$ the state of the electron is the eigenstate of s_3 with eigenvalue $\frac{1}{2}\hbar$. Calculate the probability that at later time t the electron will be measured to be in the eigenstate of s_3 with eigenvalue $\frac{1}{2}\hbar$.

14C Statistical Physics and Cosmology

(i) In equilibrium, the number density of a non-relativistic particle species is given by

$$n = g_s \left(\frac{2\pi mkT}{h^2}\right)^{3/2} e^{(\mu - mc^2)/kT}\,,$$

where m is the mass, μ is the chemical potential and g_s is the spin degeneracy. At around $t = 100$ seconds, deuterium D forms through the nuclear fusion of nonrelativistic protons p and neutrons n via the interaction:

$$p + n \leftrightarrow D\,.$$

What is the relationship between the chemical potentials of the three species when they are in chemical equilibrium? Show that the ratio of their number densities can be expressed as

$$\frac{n_D}{n_n n_p} \approx \left(\frac{h^2}{\pi m_p kT}\right)^{3/2} e^{B_D/kT}\,,$$

where the deuterium binding energy is $B_D = (m_n + m_p - m_D)c^2$ and you may take $g_D = 4$. Now consider the fractional densities $X_a = n_a/n_B$, where n_B is the baryon number of the universe, to re-express the ratio above in the form

$$\frac{X_D}{X_n X_p}$$

which incorporates the baryon-to-photon ratio η of the universe. [*You may assume that the photon density is* $n_\gamma = \frac{16\pi\zeta(3)}{(hc)^3}(kT)^3$.] From this expression, explain why deuterium does not form until well below the temperature $kT \approx B_D$.

(ii) The number density $n = N/V$ for a photon gas in equilibrium is given by the formula

$$n = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1}\,,$$

where ν is the photon frequency. By considering the substitution $x = h\nu/kT$, show that the photon number density can be expressed in the form

$$n = \alpha T^3\,,$$

where the constant α need not be evaluated explicitly.

State the equation of state for a photon gas and explain why the chemical potential of the photon vanishes. Assuming that the photon energy density $\epsilon = E/V = (4\sigma/c)T^4$, use the first law $dE = TdS - PdV + \mu dN$ to show that the entropy density is given by

$$s = S/V = \frac{16\sigma}{3c}T^3\,.$$

Hence explain why, when photons are in equilibrium at early times in our universe, their temperature varies inversely with the scale factor: $T \propto a^{-1}$.

15E Symmetries and Groups in Physics

(i) Show that the character of an $SU(2)$ transformation in the $2l+1$ dimensional irreducible representation d_l is given by

$$\chi_l(\theta) \;=\; \frac{\sin\,[(l+1/2)\theta]}{\sin\,[\theta/2]}\;.$$

What are the characters of irreducible $SO(3)$ representations?

(ii) The isospin representation of two-particle states of pions and nucleons is spanned by the basis $T = \{|\pi^+p\rangle, |\pi^+n\rangle, |\pi^0p\rangle, |\pi^0n\rangle, |\pi^-p\rangle, |\pi^-n\rangle\}$.

Pions form an isospin triplet with $\pi^+ = |1,1\rangle$, $\pi^0 = |1,0\rangle$, $\pi^- = |1,-1\rangle$; and nucleons form an isospin doublet with $p = |1/2, 1/2\rangle$, $n = |1/2, -1/2\rangle$. Find the values of the isospin for the irreducible representations into which T will decompose.

Using $I_-|j,m\rangle = \sqrt{(j-m+1)(j+m)}\,|j,m-1\rangle$, write the states of the basis T in terms of isospin states.

Consider the transitions

$$\begin{aligned} \pi^+p &\;\to\; \pi^+p \\ \pi^-p &\;\to\; \pi^-p \\ \pi^-p &\;\to\; \pi^0n \end{aligned}$$

and show that their amplitudes satisfy a linear relation.

16A Transport Processes

(i) Viscous, incompressible fluid of viscosity μ flows steadily in the x-direction in a uniform channel $0 < y < h$. The plane $y = 0$ is fixed and the plane $y = h$ has constant x-velocity U. Neglecting gravity, derive from first principles the equations of motion of the fluid and show that the x-component of the fluid velocity is $u(y)$ and satisfies

$$0 = -P_x + \mu u_{yy}, \tag{1}$$

where $P(x)$ is the pressure in the fluid. Write down the boundary conditions on u. Hence show that the volume flow rate $Q = \int_0^h u\, dy$ is given by

$$Q = \frac{Uh}{2} - \frac{P_x h^3}{12\mu}. \tag{2}$$

(ii) A heavy rectangular body of width L and infinite length (in the z-direction) is pivoted about one edge at $(x, y) = (0, 0)$ above a fixed rigid horizontal plane $y = 0$. The body has weight W per unit length in the z-direction, its centre of mass is distance $L/2$ from the pivot, and it is falling under gravity towards the fixed plane through a viscous, incompressible fluid. Let $\alpha(t) \ll 1$ be the angle between the body and the plane. Explain the approximations of lubrication theory which permit equations (1) and (2) of Part (i) to apply to the flow in the gap between the two surfaces.

Deduce that, in the gap,

$$P_x = \frac{6\mu\dot{\alpha}}{x\alpha^3},$$

where $\dot{\alpha} = d\alpha/dt$. By taking moments about $(x, y) = (0, 0)$, deduce that $\alpha(t)$ is given by

$$\frac{1}{\alpha^2} - \frac{1}{\alpha_0^2} = \frac{2Wt}{3\mu L},$$

where $\alpha(0) = \alpha_0$.

17A Mathematical Methods

(i) Give a brief description of the method of matched asymptotic expansions, as applied to a differential equation of the type

$$\epsilon y'' + Ky' + f(y) = 0, \quad 0 < x < 1,$$

where $0 < \epsilon \ll 1$, K is a non-zero constant, f is a suitable smooth function and the boundary values $y(0), y(1)$ are specified. An outline of Van Dyke's asymptotic matching principle should be included.

(ii) Consider the boundary-value problem

$$\epsilon y'' + y' - (2x+1)y = 0, \qquad y(0) = 0, \qquad y(1) = e^2$$

with $0 < \epsilon \ll 1$. Find the integrating factor for the leading-order outer problem. Hence obtain the first two terms in the outer expansion.

Rewrite the problem using an appropriate stretched inner variable. Hence obtain the first two terms of the inner exansion.

Use van Dyke's matching principle to determine all the constants. Hence show that $y'(0) = \epsilon^{-1} + \frac{25}{3} + O(\epsilon)$.

18B Nonlinear Waves

(i) Let $\Phi^+(t)$ and $\Phi^-(t)$ denote the boundary values of functions which are analytic inside and outside the unit disc centred on the origin, respectively. Let C denote the boundary of this disc. Suppose that $\Phi^+(t)$ and $\Phi^-(t)$ satisfy the jump condition

$$\Phi^+(t) = t^{-2}\Phi^-(t) + t^{-1} + \alpha(t^{-1} + t - t^{-3}), \quad t \in C,$$

where α is a constant.

Find the canonical solution of the associated homogeneous Riemann-Hilbert problem. Write down the orthogonality conditions.

(ii) Consider the linear singular integral equation

$$(t+t^{-1})\psi(t) + \frac{t-t^{-1}}{\pi i}\oint_C \frac{\psi(\tau)}{\tau - t}d\tau = 2 + 2\alpha(1 + t^2 - t^{-2}), \qquad (*)$$

where $\oint$ denotes the principal value integral.

Show that the associated Riemann-Hilbert problem has the jump condition defined in Part (i) above. Using this fact, find the value of the constant α that allows equation $(*)$ to have a solution. For this particular value of α find the unique solution $\psi(t)$.

19D Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0\,,$$

with the initial condition $u(x, 0) = \phi(x), 0 \leqslant x \leqslant 1$, and with zero boundary conditions at $x = 0$ and $x = 1$, can be solved by the method

$$u_m^{n+1} = u_m^n + \mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n), \quad m = 1, 2, \ldots, M, \quad n \geqslant 0\,,$$

where $\Delta x = 1/(M+1), \mu = \Delta t/(\Delta x)^2$, and $u_m^n \approx u(m\Delta x, n\Delta t)$. Prove that $\mu \leqslant \frac{1}{2}$ implies convergence.

(ii) By discretizing the same equation and employing the same notation as in Part (i), determine conditions on $\mu > 0$ such that the method

$$\begin{aligned}
&\left(\frac{1}{12} - \frac{1}{2}\mu\right) u_{m-1}^{n+1} + \left(\frac{5}{6} + \mu\right) u_m^{n+1} + \left(\frac{1}{12} - \frac{1}{2}\mu\right) u_{m+1}^{n+1} = \\
&\left(\frac{1}{12} + \frac{1}{2}\mu\right) u_{m-1}^{n} + \left(\frac{5}{6} - \mu\right) u_m^{n} + \left(\frac{1}{12} + \frac{1}{2}\mu\right) u_{m+1}^{n}
\end{aligned}$$

is stable.