UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Wednesday 2 June 2004 9 to 12

PAPER 2

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**.

Additional credit will be given for a substantially complete answer to either Part.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **13E, 14E** *should be in one bundle and* **3F, 8F** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1I Markov Chains

(i) Let J be a proper subset of the finite state space I of an irreducible Markov chain (X_n), whose transition matrix P is partitioned as

$$P = \begin{array}{cc} & \begin{array}{cc} J & J^c \end{array} \\ \begin{array}{c} J \\ J^c \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}.$$

If only visits to states in J are recorded, we see a J-valued Markov chain $(\tilde{X}_n)$; show that its transition matrix is

$$\tilde{P} = A + B \sum_{n \geqslant 0} D^n C = A + B(I-D)^{-1} C\,.$$

(ii) Local MP Phil Anderer spends his time in London in the Commons (C), in his flat (F), in the bar (B) or with his girlfriend (G). Each hour, he moves from one to another according to the transition matrix P, though his wife (who knows nothing of his girlfriend) believes that his movements are governed by transition matrix P^W:

$$P = \begin{array}{cc} & \begin{array}{cccc} C & F & B & G \end{array} \\ \begin{array}{c} C \\ F \\ B \\ G \end{array} & \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix} \end{array} \qquad P^W = \begin{array}{cc} & \begin{array}{ccc} C & F & B \end{array} \\ \begin{array}{c} C \\ F \\ B \end{array} & \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

The public only sees Phil when he is in $J = \{C, F, B\}$; calculate the transition matrix $\tilde{P}$ which they believe controls his movements.

Each time the public Phil moves to a new location, he phones his wife; write down the transition matrix which governs the sequence of locations from which the public Phil phones, and calculate its invariant distribution.

Phil's wife notes down the location of each of his calls, and is getting suspicious – he is not at his flat often enough. Confronted, Phil swears his fidelity and resolves to dump his troublesome transition matrix, choosing instead

$$P^* = \begin{array}{cc} & \begin{array}{cccc} C & F & B & G \end{array} \\ \begin{array}{c} C \\ F \\ B \\ G \end{array} & \begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 0 & 3/8 & 1/8 & 1/2 \\ 2/10 & 1/10 & 1/10 & 6/10 \end{pmatrix} \end{array}$$

Will this deal with his wife's suspicions? Explain your answer.

2B Principles of Dynamics

(i) Consider a light rigid circular wire of radius a and centre O. The wire lies in a vertical plane, which rotates about the vertical axis through O. At time t the plane containing the wire makes an angle $\phi(t)$ with a fixed vertical plane. A bead of mass m is threaded onto the wire. The bead slides without friction along the wire, and its location is denoted by A. The angle between the line OA and the downward vertical is $\theta(t)$.

Show that the Lagrangian of the system is

$$\frac{ma^2}{2}\dot{\theta}^2 + \frac{ma^2}{2}\dot{\phi}^2 \sin^2\theta + mga\cos\theta \ .$$

Calculate two independent constants of the motion, and explain their physical significance.

(ii) A dynamical system has Hamiltonian $H(q,p,\lambda)$, where λ is a parameter. Consider an ensemble of identical systems chosen so that the number density of systems, $f(q,p,t)$, in the phase space element $dq\,dp$ is either zero or one. Prove Liouville's Theorem, namely that the total area of phase space occupied by the ensemble is time-independent.

Now consider a single system undergoing periodic motion $q(t), p(t)$. Give a heuristic argument based on Liouville's Theorem to show that the area enclosed by the orbit,

$$I = \oint p\,dq\,,$$

is approximately conserved as the parameter λ is slowly varied (i.e. that I is an adiabatic invariant).

Consider $H(q,p,\lambda) = \frac{1}{2}p^2 + \lambda q^{2n}$, with n a positive integer. Show that as λ is slowly varied the energy of the system, E, varies as

$$E \propto \lambda^{1/(n+1)}\,.$$

3F Functional Analysis

(i) Prove Riesz's Lemma, that if V is a normed space and A is a vector subspace of V such that for some $0 \leqslant k < 1$ we have $d(x, A) \leqslant k$ for all $x \in V$ with $||x|| = 1$, then A is dense in V. [Here $d(x, A)$ denotes the distance from x to A.]

Deduce that any normed space whose unit ball is compact is finite-dimensional. [*You may assume that every finite-dimensional normed space is complete.*]

Give an example of a sequence $f_1, f_2, \ldots$ in an infinite-dimensional normed space such that $||f_n|| \leqslant 1$ for all n, but $f_1, f_2, \ldots$ has no convergent subsequence.

(ii) Let V be a vector space, and let $||.||_1$ and $||.||_2$ be two norms on V. What does it mean to say that $||.||_1$ and $||.||_2$ are *equivalent*?

Show that on a finite-dimensional vector space all norms are equivalent. Deduce that every finite-dimensional normed space is complete.

Exhibit two norms on the vector space l^1 that are not equivalent.

In addition, exhibit two norms on the vector space l^∞ that are not equivalent.

4G Groups, Rings and Fields

(i) State Gauss' Lemma on polynomial irreducibility. State and prove Eisenstein's criterion.

(ii) Which of the following polynomials are irreducible over $\mathbb{Q}$? Justify your answers.

(a) $x^7 - 3x^3 + 18x + 12$

(b) $x^4 - 4x^3 + 11x^2 - 3x - 5$

(c) $1 + x + x^2 + \ldots + x^{p-1}$ with p prime

[*Hint: consider substituting $y = x - 1$.*]

(d) $x^n + px + p^2$ with p prime.

[*Hint: show any factor has degree at least two, and consider powers of p dividing coefficients.*]

5C Electromagnetism

(i) Write down the general solution of Poisson's equation. Derive from Maxwell's equations the Biot-Savart law for the magnetic field of a steady localised current distribution.

(ii) A plane rectangular loop with sides of length a and b lies in the plane $z = 0$ and is centred on the origin. Show that when $r = |\mathbf{r}| \gg a, b$, the vector potential $\mathbf{A}(\mathbf{r})$ is given approximately by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \wedge \mathbf{r}}{r^3},$$

where $\mathbf{m} = Iab\hat{\mathbf{z}}$ is the magnetic moment of the loop.

Hence show that the magnetic field $\mathbf{B}(\mathbf{r})$ at a great distance from an arbitrary small plane loop of area A, situated in the xy-plane near the origin and carrying a current I, is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 IA}{4\pi r^5}\big(3xz,\, 3yz,\, 2r^2 - 3x^2 - 3y^2\big).$$

6B Nonlinear Dynamical Systems

(i) A linear system in $\mathbb{R}^2$ takes the form $\dot{\mathbf{x}} = \mathsf{A}\mathbf{x}$. Explain (without detailed calculation but by giving examples) how to classify the dynamics of the system in terms of the determinant and the trace of A. Show your classification graphically, and describe the dynamics that occurs on the boundaries of the different regions on your diagram.

(ii) A nonlinear system in $\mathbb{R}^2$ has the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f}(0) = 0$. The Jacobian (linearization) A of $\mathbf{f}$ at the origin is non-hyperbolic, with one eigenvalue of A in the left-hand half-plane. Define the *centre manifold* for this system, and explain (stating carefully any results you use) how the dynamics near the origin may be reduced to a one-dimensional system on the centre manifold.

A dynamical system of this type has the form

$$\begin{aligned}\dot{x} &= ax^3 + bxy + cx^5 + dx^3y + exy^2 + fx^7 + gx^5y \\ \dot{y} &= -y + x^2 - x^4\end{aligned}$$

Find the coefficients for the expansion of the centre manifold correct up to and including terms of order x^6, and write down in terms of these coefficients the equation for the dynamics on the centre manifold up to order x^7. Using this reduced equation, give a complete set of conditions on the coefficients $a, b, c, \ldots$ that guarantee that the origin is stable.

7G Geometry of Surfaces

(i) What is a *geodesic* on a surface M with Riemannian metric, and what are *geodesic polar co-ordinates centred at a point* P on M? State, without proof, formulae for the Riemannian metric and the Gaussian curvature in terms of geodesic polar co-ordinates.

(ii) Show that a surface with constant Gaussian curvature 0 is locally isometric to the Euclidean plane.

8F Graph Theory

(i) State a result of Euler, relating the number of vertices, edges and faces of a plane graph. Show that if G is a plane graph then $\chi(G) \leq 5$.

(ii) Define the chromatic polynomial $p_G(t)$ of a graph G. Show that

$$p_G(t) = t^n - a_1 t^{n-1} + a_2 t^{n-2} + \ldots + (-1)^n a_n$$

where $a_1, \ldots, a_n$ are non-negative integers. Explain, with proof, how the chromatic polynomial is related to the number of vertices, edges and triangles in G. Show that if C_n is a cycle of length $n \geq 3$, then

$$p_{C_n}(t) = (t-1)^n + (-1)^n (t-1).$$

9H Coding and Cryptography

(i) Describe how a stream cypher operates. What is a one-time pad?

A one-time pad is used to send the message $x_1x_2x_3x_4x_5x_6y_7$ which is encoded as 0101011. By mistake, it is reused to send the message $y_0x_1x_2x_3x_4x_5x_6$ which is encoded as 0100010. Show that $x_1x_2x_3x_4x_5x_6$ is one of two possible messages, and find the two possibilities.

(ii) Describe the RSA system associated with a public key e, a private key d and the product N of two large primes.

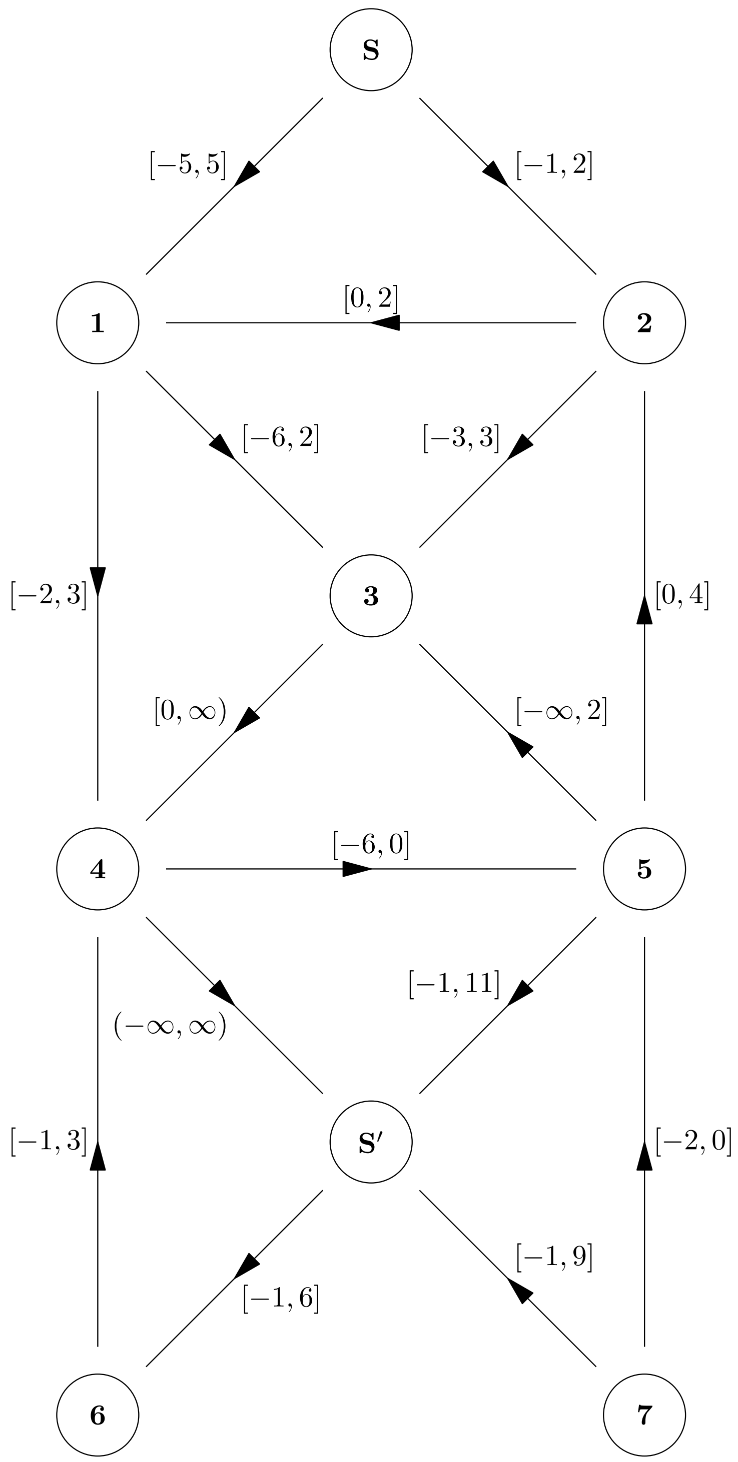
Give a simple example of how the system is vulnerable to a homomorphism attack. Explain how a signature system prevents such an attack. [*You are not asked to give an explicit signature system.*]

Explain how to factorise N when e, d and N are known.

10J Algorithms and Networks

(i) Define the minimum path and the maximum tension problems for a network with span intervals specified for each arc. State without proof the connection between the two problems, and describe the Max Tension Min Path algorithm of solving them.

(ii) Find the minimum path between nodes **S** and **S′** in the network below. The span intervals are displayed alongside the arcs.



11J Principles of Statistics

(i) In the context of a decision-theoretic approach to statistics, what is a *loss function*? a *decision rule*? the *risk function* of a decision rule? the *Bayes risk* of a decision rule? the *Bayes rule* with respect to a given prior distribution?

Show how the Bayes rule with respect to a given prior distribution is computed.

(ii) A sample of n people is to be tested for the presence of a certain condition. A single real-valued observation is made on each one; this observation comes from density f_0 if the condition is absent, and from density f_1 if the condition is present. Suppose $\theta_i = 0$ if the i^{th} person does not have the condition, $\theta_i = 1$ otherwise, and suppose that the prior distribution for the θ_i is that they are independent with common distribution $P(\theta_i = 1) = p \in (0,1)$, where p is known. If X_i denotes the observation made on the i^{th} person, what is the posterior distribution of the θ_i?

Now suppose that the loss function is defined by

$$L_0(\theta, a) \equiv \sum_{j=1}^{n} (\alpha a_j (1-\theta_j) + \beta(1-a_j)\theta_j)$$

for action $a \in [0,1]^n$, where α, β are positive constants. If π_j denotes the posterior probability that $\theta_j = 1$ given the data, prove that the Bayes rule for this prior and this loss function is to take $a_j = 1$ if π_j exceeds the threshold value $\alpha/(\alpha+\beta)$, and otherwise to take $a_j = 0$.

In an attempt to control the proportion of false positives, it is proposed to use a different loss function, namely,

$$L_1(\theta, a) \equiv L_0(\theta, a) + \gamma I_{\{\sum a_j > 0\}} \left(1 - \frac{\sum \theta_j a_j}{\sum a_j}\right),$$

where $\gamma > 0$. Prove that the Bayes rule is once again a threshold rule, that is, we take action $a_j = 1$ if and only if $\pi_j > \lambda$, and determine λ as fully as you can.

12I Computational Statistics and Statistical Modelling

(i) Suppose we have independent observations $Y_1, \ldots, Y_n$, and we assume that for $i = 1, \ldots, n$, Y_i is Poisson with mean μ_i, and $\log(\mu_i) = \beta^T x_i$, where $x_1, \ldots, x_n$ are given covariate vectors each of dimension p, where β is an unknown vector of dimension p, and $p < n$. Assuming that $\{x_1, \ldots, x_n\}$ span $\mathbb{R}^p$, find the equation for $\hat{\beta}$, the maximum likelihood estimator of β, and write down the large-sample distribution of $\hat{\beta}$.

(ii) A long-term agricultural experiment had 90 grassland plots, each 25m $\times$ 25m, differing in biomass, soil pH, and species richness (the count of species in the whole plot). While it was well-known that species richness declines with increasing biomass, it was not known how this relationship depends on soil pH, which for the given study has possible values "low", "medium" or "high", each taken 30 times. Explain the commands input, and interpret the resulting output in the (slightly edited) *R* output below, in which "species" represents the species count.

(The first and last 2 lines of the data are reproduced here as an aid. You may assume that the factor pH has been correctly set up.)

```
> species
      pH    Biomass Species
1  high 0.46929722      30
2  high 1.73087043      39
  ......................
  ......................
89  low 4.36454121       7
90  low 4.87050789       3

>  summary(glm(Species ~Biomass, family = poisson))
Call:
glm(formula = Species ~ Biomass, family = poisson)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  3.184094   0.039159   81.31  < 2e-16
Biomass     -0.064441   0.009838   -6.55 5.74e-11

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 452.35  on 89  degrees of freedom
Residual deviance: 407.67  on 88  degrees of freedom

Number of Fisher Scoring iterations: 4

> summary(glm(Species ~pH*Biomass, family = poisson))
Call:
glm(formula = Species ~ pH * Biomass, family = poisson)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)    3.76812    0.06153  61.240  < 2e-16
pHlow         -0.81557    0.10284  -7.931 2.18e-15
```

Question continues on next page.

```
pHmid         -0.33146    0.09217  -3.596 0.000323
Biomass       -0.10713    0.01249  -8.577  < 2e-16
pHlow:Biomass -0.15503    0.04003  -3.873 0.000108
pHmid:Biomass -0.03189    0.02308  -1.382 0.166954

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 452.346  on 89  degrees of freedom
Residual deviance:  83.201  on 84  degrees of freedom

Number of Fisher Scoring iterations: 4
```

13E Foundations of Quantum Mechanics

(i) The creation and annihilation operators for a harmonic oscillator of angular frequency ω satisfy the commutation relation $[a, a^\dagger] = 1$. Write down an expression for the Hamiltonian H in terms of a and $a^\dagger$.

There exists a unique ground state $|0\rangle$ of H such that $a|0\rangle = 0$. Explain how the space of eigenstates $|n\rangle$, $n = 0, 1, 2, \dots$ of H is formed, and deduce the eigenenergies for these states. Show that

$$a|n\rangle = \sqrt{n}|n-1\rangle \,, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \,.$$

(ii) Write down the number operator N of the harmonic oscillator in terms of a and $a^\dagger$. Show that

$$N|n\rangle = n|n\rangle \,.$$

The operator K_r is defined to be

$$K_r = \frac{a^{\dagger\,r} a^r}{r!} \,, \qquad r = 0, 1, 2, \dots .$$

Show that K_r commutes with N. Show also that

$$K_r|n\rangle = \begin{cases} \frac{n!}{(n-r)!\,r!}|n\rangle & r \le n \,, \\ 0 & r > n \,. \end{cases}$$

By considering the action of K_r on the state $|n\rangle$ show that

$$\sum_{r=0}^{\infty} (-1)^r K_r = |0\rangle\langle 0| \,.$$

14E Quantum Physics

(i) A simple model of a crystal consists of an infinite linear array of sites equally spaced with separation b. The probability amplitude for an electron to be at the n-th site is c_n, $n = 0, \pm 1, \pm 2, \ldots$. The Schrödinger equation for the $\{c_n\}$ is

$$Ec_n \;=\; E_0 c_n - A(c_{n-1} + c_{n+1}) \;,$$

where A is real and positive. Show that the allowed energies E of the electron must lie in a band $|E - E_0| \leq 2A$, and that the dispersion relation for E written in terms of a certain parameter k is given by

$$E \;=\; E_0 - 2A \cos kb \;.$$

What is the physical interpretation of E_0, A and k?

(ii) Explain briefly the idea of group velocity and show that it is given by

$$v \;=\; \frac{1}{\hbar}\frac{dE(k)}{dk} \;,$$

for an electron of momentum $\hbar k$ and energy $E(k)$.

An electron of charge q confined to one dimension moves in a periodic potential under the influence of an electric field $\mathcal{E}$. Show that the equation of motion for the electron is

$$\dot{v} \;=\; \frac{q\mathcal{E}}{\hbar^2}\frac{d^2E}{dk^2} \;,$$

where $v(t)$ is the group velocity of the electron at time t. Explain why

$$m^* \;=\; \hbar^2 \left(\frac{d^2E}{dk^2}\right)^{-1}$$

can be interpreted as an effective mass.

Show briefly how the absence from a band of an electron of charge q and effective mass $m^* < 0$ can be interpreted as the presence of a 'hole' carrier of charge $-q$ and effective mass $-m^*$.

In the model of Part (i) show that

(a) for $k^2 \ll 12/b^2$ an electron behaves like a free particle of mass $\hbar^2/(2Ab^2)$;

(b) for $(\pi/b - k)^2 \ll 12/b^2$ a hole behaves like a free particle of mass $\hbar^2/(2Ab^2)$.

15C General Relativity

(i) State and prove Birkhoff's theorem.

(ii) Derive the Schwarzschild metric and discuss its relevance to the problem of gravitational collapse and the formation of black holes.

[*Hint: You may assume that the metric takes the form*

$$ds^2 = -e^{\nu(r,t)}\,dt^2 + e^{\lambda(r,t)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

and that the non-vanishing components of the Einstein tensor are given by

$$G_{tt} = \frac{e^{2\nu+\lambda}}{r^2}(-1 + e^{\lambda} + r\lambda'), \quad G_{rt} = e^{(\nu+\lambda)/2}\frac{\dot\lambda}{r}, \quad G_{rr} = \frac{e^{\lambda}}{r^2}(1 - e^{-\lambda} + r\nu'),$$

$$G_{\theta\theta} = \tfrac{1}{4}r^2e^{-\lambda}\Big[2\nu'' + (\nu')^2 + \frac{2}{r}(\nu' - \lambda') - \nu'\lambda'\Big] - \tfrac{1}{4}r^2e^{-\nu}\big[2\ddot\lambda + (\dot\lambda)^2 - \dot\lambda\dot\nu\big],$$

$G_{tr} = G_{rt}$ *and* $G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta}$.]

16A Theoretical Geophysics

(i) Sketch the rays in a small region near the relevant boundary produced by reflection and refraction of a P-wave incident (a) from the mantle on the core-mantle boundary, (b) from the outer core on the inner-core boundary, and (c) from the mantle on the Earth's surface. [*In each case, the region should be sufficiently small that the boundary appears to be planar.*]

Describe the ray paths denoted by SS, PcP, SKS and $PKIKP$.

Sketch the travel-time ($T-\Delta$) curves for P and PcP paths from a surface source.

(ii) From the surface of a flat Earth, an explosive source emits P-waves downwards into a stratified sequence of homogeneous horizontal elastic layers of thicknesses $h_1, h_2, h_3, \ldots$ and P-wave speeds $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$. A line of seismometers on the surface records the travel times of the various arrivals as a function of the distance x from the source. Calculate the travel times, $T_d(x)$ and $T_r(x)$, of the direct wave and the wave that reflects exactly once at the bottom of layer 1.

Show that the travel time for the head wave that refracts in layer n is given by

$$T_n = \frac{x}{\alpha_n} + \sum_{i=1}^{n-1}\frac{2h_i}{\alpha_i}\left(1 - \frac{\alpha_i^2}{\alpha_n^2}\right)^{1/2}.$$

Sketch the travel-time curves for T_r, T_d and T_2 on a single diagram and show that T_2 is tangent to T_r.

Explain how the α_i and h_i can be constructed from the travel times of first arrivals provided that each head wave is the first arrival for some range of x.

17A Mathematical Methods

(i) Consider the integral equation

$$\phi(x) = -\lambda \int_a^b K(x,t)\phi(t)dt + g(x), \qquad (\dagger)$$

for ϕ in the interval $a \leq x \leq b$, where λ is a real parameter and $g(x)$ is given. Describe the method of successive approximations for solving ($\dagger$).

Suppose that

$$|K(x,t)| \leq M, \qquad \forall x, t \in [a,b].$$

By using the Cauchy-Schwarz inequality, or otherwise, show that the successive-approximation series for $\phi(x)$ converges absolutely provided

$$|\lambda| < \frac{1}{M(b-a)}.$$

(ii) The real function $\psi(x)$ satisfies the differential equation

$$-\psi''(x) + \lambda\psi(x) = h(x), \qquad 0 < x < 1, \qquad (\star)$$

where $h(x)$ is a given smooth function on $[0,1]$, subject to the boundary conditions

$$\psi'(0) = \psi(0), \quad \psi(1) = 0.$$

By integrating ($\star$), or otherwise, show that $\psi(x)$ obeys

$$\psi(0) = \frac{1}{2}\int_0^1 (1-t)h(t)\,dt - \frac{1}{2}\lambda \int_0^1 (1-t)\psi(t)\,dt.$$

Hence, or otherwise, deduce that $\psi(x)$ obeys an equation of the form ($\dagger$), with

$$K(x,t) = \begin{cases} \frac{1}{2}(1-x)(1+t), & 0 \leq t \leq x \leq 1, \\ \frac{1}{2}(1+x)(1-t), & 0 \leq x \leq t \leq 1, \end{cases}$$

$$\text{and} \quad g(x) = \int_0^1 K(x,t)h(t)\,dt.$$

Deduce that the series solution for $\psi(x)$ converges provided $|\lambda| < 2$.

18B Nonlinear Waves

(i) Let $u(x,t)$ satisfy the Burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},$$

where ν is a positive constant. Consider solutions of the form $u=u(X)$, where $X=x-Ut$ and U is a constant, such that

$$u\to u_2,\quad \frac{\partial u}{\partial X}\to 0\quad \text{as}\quad X\to-\infty;\qquad u\to u_1,\quad \frac{\partial u}{\partial X}\to 0\quad \text{as}\quad X\to\infty\ ,$$

with $u_2>u_1$.

Show that U satisfies the so-called shock condition

$$U=\frac{1}{2}(u_2+u_1).$$

By using the factorisation

$$\frac{1}{2}u^2-Uu+A=\frac{1}{2}(u-u_1)(u-u_2),$$

where A is the constant of integration, express u in terms of X, u_1, u_2 and ν.

(ii) According to shallow-water theory, river waves are characterised by the PDEs

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\cos\alpha\frac{\partial h}{\partial x}=g\sin\alpha-C_f\frac{v^2}{h},$$
$$\frac{\partial h}{\partial t}+v\frac{\partial h}{\partial x}+h\frac{\partial v}{\partial x}=0,$$

where $h(x,t)$ denotes the depth of the river, $v(x,t)$ denotes the mean velocity, α is the constant angle of inclination, and C_f is the constant friction coefficient.

Find the characteristic velocities and the characteristic form of the equations. Find the Riemann variables and show that if $C_f=0$ then the Riemann variables vary linearly with t on the characteristics.

19D Numerical Analysis

(i) The *five-point equations*, which are obtained when the Poisson equation $\nabla^2 u=f$ (with Dirichlet boundary conditions) is discretized in a square, are

$$-u_{m-1,n}-u_{m,n-1}-u_{m+1,n}-u_{m,n+1}+4u_{m,n}=f_{m,n},\quad m,n=1,2,\ldots,M,$$

where $u_{0,n},\ u_{M+1,n},\ u_{m,0},\ u_{m,M+1}=0$ for all $m,n=1,2,\ldots,M$.

Formulate the Gauss–Seidel method for the above linear system and prove its convergence. In the proof you should carefully state any theorems you use. [*You may use Part (ii) of this question.*]

(ii) By arranging the two-dimensional arrays $\{u_{m,n}\}_{m,n=1,\ldots,M}$ and $\{b_{m,n}\}_{m,n=1,\ldots,M}$ into the column vectors $\mathbf{u}\in\mathbb{R}^{M^2}$ and $\mathbf{b}\in\mathbb{R}^{M^2}$ respectively, the linear system described in Part (i) takes the matrix form $A\mathbf{u}=\mathbf{b}$. Prove that, regardless of the ordering of the points on the grid, the matrix A is symmetric and positive definite.