UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 3 June 2004 9 to 12

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Additional credit will be given to substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** …, **H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Let V be a finite-dimensional vector space over $\mathbb{R}$. What is the *dual space* of V? Prove that the dimension of the dual space is the same as that of V.

2F Groups, Rings and Modules

Let R be the subring of all z in $\mathbb{C}$ of the form

$$z = \frac{a + b\sqrt{-3}}{2}$$

where a and b are in $\mathbb{Z}$ and $a \equiv b \pmod{2}$. Prove that $N(z) = z\bar{z}$ is a non-negative element of $\mathbb{Z}$, for all z in R. Prove that the multiplicative group of units of R has order 6. Prove that $7R$ is the intersection of two prime ideals of R.

[*You may assume that R is a unique factorization domain.*]

3G Geometry

State Euler's formula for a convex polyhedron with F faces, E edges, and V vertices.

Show that any regular polyhedron whose faces are pentagons has the same number of vertices, edges and faces as the dodecahedron.

4F Analysis II

Let X and X' be metric spaces with metrics d and d'. If $u = (x, x')$ and $v = (y, y')$ are any two points of $X \times X'$, prove that the formula

$$D(u, v) = \max\big\{d(x, y), d'(x', y')\big\}$$

defines a metric on $X \times X'$. If $X = X'$, prove that the diagonal Δ of $X \times X$ is closed in $X \times X$.

5E Further Analysis

Let C be the contour that goes once round the boundary of the square

$$\{z : -1 \leqslant \operatorname{Re} z \leqslant 1, -1 \leqslant \operatorname{Im} z \leqslant 1\}$$

in an anticlockwise direction. What is $\displaystyle\int_C \frac{dz}{z}$? Briefly justify your answer.

Explain why the integrals along each of the four edges of the square are equal. Deduce that $\displaystyle\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}$.

6D Methods

Let

$$S[x] = \int_0^T \tfrac{1}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \mathrm{d}t\,, \qquad x(0) = a\,, \quad x(T) = b\,.$$

For any variation $\delta x(t)$ with $\delta x(0) = \delta x(T) = 0$, show that $\delta S = 0$ when $x = x_c$ with

$$x_c(t) = \frac{1}{\sin \omega T}\Big[a \sin \omega(T-t) + b \sin \omega t\Big]\,.$$

By using integration by parts, show that

$$S[x_c] = \left[\tfrac{1}{2} x_c \dot{x}_c\right]_0^T = \frac{\omega}{2 \sin \omega T}\Big[(a^2 + b^2)\cos \omega T - 2ab\Big]\,.$$

7B Electromagnetism

A wire is bent into the shape of three sides of a rectangle and is held fixed in the $z = 0$ plane, with base $x = 0$ and $-\ell < y < \ell$, and with arms $y = \pm \ell$ and $0 < x < \ell$. A second wire moves smoothly along the arms: $x = X(t)$ and $-\ell < y < \ell$ with $0 < X < \ell$. The two wires have resistance R per unit length and mass M per unit length. There is a time-varying magnetic field $B(t)$ in the z-direction.

Using the law of induction, find the electromotive force around the circuit made by the two wires.

Using the Lorentz force, derive the equation

$$M\ddot{X} = -\frac{B}{R(X + 2\ell)} \frac{d}{dt}(X \ell B)\,.$$

8B Special Relativity

Write down the Lorentz transformation with one space dimension between two inertial frames S and S' moving relatively to one another at speed V.

A particle moves at velocity u in frame S. Find its velocity u' in frame S' and show that u' is always less than c.

9D Quantum Mechanics

Write down the expressions for the classical energy and angular momentum for an electron in a hydrogen atom. In the Bohr model the angular momentum L is quantised so that

$$L = n\hbar\,,$$

for integer n. Assuming circular orbits, show that the radius of the n'th orbit is

$$r_n = n^2 a\,,$$

and determine a. Show that the corresponding energy is then

$$E_n = -\frac{e^2}{8\pi\epsilon_0\, r_n}\,.$$

10C Fluid Dynamics

State Bernoulli's equation for unsteady motion of an irrotational, incompressible, inviscid fluid subject to a conservative body force $-\nabla\chi$.

A long vertical U-tube of uniform cross section contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation

$$h\frac{d^2\zeta}{dt^2} + g\zeta = 0$$

governing the displacement ζ of the surface on one side of the U-tube, where t is time and g is the acceleration due to gravity.

11A Numerical Analysis

The linear system

$$\begin{bmatrix} \alpha & 2 & 1 \\ 1 & \alpha & 2 \\ 2 & 1 & \alpha \end{bmatrix} \mathbf{x} = \mathbf{b},$$

where real $\alpha \neq 0$ and $\mathbf{b} \in \mathbb{R}^3$ are given, is solved by the iterative procedure

$$\mathbf{x}^{(k+1)} = -\frac{1}{\alpha} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \frac{1}{\alpha}\mathbf{b}, \qquad k \geqslant 0.$$

Determine the conditions on α that guarantee convergence.

12G Optimization

Consider the two-person zero-sum game Rock, Scissors, Paper. That is, a player gets 1 point by playing Rock when the other player chooses Scissors, or by playing Scissors against Paper, or Paper against Rock; the losing player gets -1 point. Zero points are received if both players make the same move.

Suppose player one chooses Rock and Scissors (but never Paper) with probabilities p and $1-p$, $0 \leqslant p \leqslant 1$. Write down the maximization problem for player two's optimal strategy. Determine the optimal strategy for each value of p.

SECTION II

13E Linear Algebra

(i) Let V be an n-dimensional vector space over $\mathbb{C}$ and let $\alpha : V \to V$ be an endomorphism. Suppose that the characteristic polynomial of α is $\Pi_{i=1}^{k}(x-\lambda_i)^{n_i}$, where the λ_i are distinct and $n_i > 0$ for every i.

Describe all possibilities for the minimal polynomial and prove that there are no further ones.

(ii) Give an example of a matrix for which both the characteristic and the minimal polynomial are $(x-1)^3(x-3)$.

(iii) Give an example of two matrices A, B with the same rank and the same minimal and characteristic polynomials such that there is no invertible matrix P with $PAP^{-1} = B$.

14F Groups, Rings and Modules

Let L be the group $\mathbb{Z}^3$ consisting of 3-dimensional row vectors with integer components. Let M be the subgroup of L generated by the three vectors

$$u = (1,2,3),\ \ v = (2,3,1),\ \ w = (3,1,2)\,.$$

(i) What is the index of M in L?

(ii) Prove that M is not a direct summand of L.

(iii) Is the subgroup N generated by u and v a direct summand of L?

(iv) What is the structure of the quotient group L/M?

15G Geometry

Let a, b, c be the lengths of a right-angled triangle in spherical geometry, where c is the hypotenuse. Prove the Pythagorean theorem for spherical geometry in the form

$$\cos c = \cos a \cos b\,.$$

Now consider such a spherical triangle with the sides a, b replaced by $\lambda a, \lambda b$ for a positive number λ. Show that the above formula approaches the usual Pythagorean theorem as λ approaches zero.

16F Analysis II

State and prove the contraction mapping theorem.

Let a be a positive real number, and take $X = [\sqrt{\frac{a}{2}},\ \infty)$. Prove that the function

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

is a contraction from X to X. Find the unique fixed point of f.

17E Further Analysis

(i) Explain why the formula

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

defines a function that is analytic on the domain $\mathbb{C} \setminus \mathbb{Z}$. [You need not give full details, but should indicate what results are used.]

Show also that $f(z+1) = f(z)$ for every z such that $f(z)$ is defined.

(ii) Write $\log z$ for $\log r + i\theta$ whenever $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leqslant \pi$. Let g be defined by the formula

$$g(z) = f\left(\frac{1}{2\pi i}\log z\right) .$$

Prove that g is analytic on $\mathbb{C} \setminus \{0, 1\}$.

[*Hint: What would be the effect of redefining* $\log z$ *to be* $\log r + i\theta$ *when* $z = re^{i\theta}$, $r > 0$ *and* $0 \leqslant \theta < 2\pi$?]

(iii) Determine the nature of the singularity of g at $z = 1$.

18D Methods

Starting from the Euler–Lagrange equations, show that the condition for the variation of the integral $\int I(y,y')\,\mathrm{d}x$ to be stationary is

$$I - y'\frac{\partial I}{\partial y'} = \text{constant}\,.$$

In a medium with speed of light $c(y)$ the ray path taken by a light signal between two points satisfies the condition that the time taken is stationary. Consider the region $0 < y < \infty$ and suppose $c(y) = e^{\lambda y}$. Derive the equation for the light ray path $y(x)$. Obtain the solution of this equation and show that the light ray between $(-a,0)$ and $(a,0)$ is given by

$$e^{\lambda y} = \frac{\cos\lambda x}{\cos\lambda a}\,,$$

if $\lambda a < \frac{\pi}{2}$.

Sketch the path for λa close to $\frac{\pi}{2}$ and evaluate the time taken for a light signal between these points.

[*The substitution $u = k\,e^{\lambda y}$, for some constant k, should prove useful in solving the differential equation.*]

19B Electromagnetism

Starting from Maxwell's equations, derive the law of energy conservation in the form

$$\frac{\partial W}{\partial t} + \nabla\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0,$$

where $W = \dfrac{\epsilon_0}{2}E^2 + \dfrac{1}{2\mu_0}B^2$ and $\mathbf{S} = \dfrac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$.

Evaluate W and $\mathbf{S}$ for the plane electromagnetic wave in vacuum

$$\mathbf{E} = (E_0\cos(kz-\omega t), 0, 0) \qquad \mathbf{B} = (0, B_0\cos(kz-\omega t), 0),$$

where the relationships between E_0, B_0, ω and k should be determined. Show that the electromagnetic energy propagates at speed $c^2 = 1/(\epsilon_0\mu_0)$, i.e. show that $S = Wc$.

20D Quantum Mechanics

A one-dimensional system has the potential

$$V(x) = \begin{cases} 0 & x < 0\,, \\ \frac{\hbar^2 U}{2m} & 0 < x < L\,, \\ 0 & x > L\,. \end{cases}$$

For energy $E = \hbar^2 \epsilon/(2m)$, $\epsilon < U$, the wave function has the form

$$\psi(x) = \begin{cases} a\, e^{ikx} + c\, e^{-ikx} & x < 0\,, \\ e\, \cosh Kx + f\, \sinh Kx & 0 < x < L\,, \\ d\, e^{ik(x-L)} + b\, e^{-ik(x-L)} & x > L\,. \end{cases}$$

By considering the relation between incoming and outgoing waves explain why we should expect

$$|c|^2 + |d|^2 = |a|^2 + |b|^2\,.$$

Find four linear relations between a, b, c, d, e, f. Eliminate d, e, f and show that

$$c = \frac{1}{D}\left[b + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big) \sinh KL\; a \right]\,,$$

where $D = \cosh KL - \frac{1}{2}\big(\lambda + \frac{1}{\lambda}\big) \sinh KL$ and $\lambda = K/(ik)$. By using the result for c, or otherwise, explain why the solution for d is

$$d = \frac{1}{D}\left[a + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big) \sinh KL\; b \right]\,.$$

For $b = 0$ define the transmission coefficient T and show that, for large L,

$$T \approx 16\, \frac{\epsilon(U-\epsilon)}{U^2}\, e^{-2\sqrt{U-\epsilon}\, L}\,.$$

21C Fluid Dynamics

Use separation of variables to determine the irrotational, incompressible flow

$$\mathbf{u} = U\frac{a^3}{r^3}\left(\cos\theta\,\mathbf{e}_r + \tfrac{1}{2}\sin\theta\,\mathbf{e}_\theta\right)$$

around a solid sphere of radius a translating at velocity U along the direction $\theta = 0$ in spherical polar coordinates r and θ.

Show that the total kinetic energy of the fluid is

$$K = \tfrac{1}{4}M_f U^2,$$

where M_f is the mass of fluid displaced by the sphere.

A heavy sphere of mass M is released from rest in an inviscid fluid. Determine its speed after it has fallen through a distance h in terms of M, M_f, g and h.

22A Numerical Analysis

Given $f \in C^3[0,1]$, we approximate $f'(\frac{1}{3})$ by the linear combination

$$\mathcal{T}[f] = -\frac{5}{3}f(0) + \frac{4}{3}f(\tfrac{1}{2}) + \frac{1}{3}f(1)\,.$$

By finding the Peano kernel, determine the least constant c such that

$$\left|\mathcal{T}[f] - f'(\tfrac{1}{3})\right| \le c\left\|f^{'''}\right\|_\infty.$$

23G Optimization

Consider the following linear programming problem:

$$\begin{aligned} \text{maximize} \quad & -x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geqslant 3, \\ & -x_1 + 2x_2 \geqslant 6, \\ & -x_1 + x_2 \leqslant 2, \\ & x_2 \leqslant 5, \\ & x_i \geqslant 0, \quad i = 1,2. \end{aligned}$$

Write down the Phase One problem in this case, and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve the above maximization problem. That is, find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.