UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 1st June 2004 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **C** *and* **D** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Algebra and Geometry

State Lagrange's Theorem.

Show that there are precisely two non-isomorphic groups of order 10. [You may assume that a group whose elements are all of order 1 or 2 has order 2^k.]

2D Algebra and Geometry

Define the Möbius group, and describe how it acts on $\mathbb{C} \cup \{\infty\}$.

Show that the subgroup of the Möbius group consisting of transformations which fix 0 and ∞ is isomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Now show that the subgroup of the Möbius group consisting of transformations which fix 0 and 1 is also isomorphic to $\mathbb{C}^*$.

3C Vector Calculus

If $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields, show that

(i) $\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - \mathbf{G}(\boldsymbol{\nabla} \cdot \mathbf{F}) + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}\,,$

(ii) $\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F})\,.$

4C Vector Calculus

Define the curvature, κ, of a curve in $\mathbb{R}^3$.

The curve C is parametrised by

$$\mathbf{x}(t) = \left(\frac{1}{2}e^t \cos t,\, \frac{1}{2}e^t \sin t,\, \frac{1}{\sqrt{2}}e^t\right) \quad \text{for } -\infty < t < \infty\,.$$

Obtain a parametrisation of the curve in terms of its arc length, s, measured from the origin. Hence obtain its curvature, $\kappa(s)$, as a function of s.

SECTION II

5D Algebra and Geometry

Let $G = \langle g, h \mid h^2 = 1, g^6 = 1, hgh^{-1} = g^{-1} \rangle$ be the dihedral group of order 12.

i) List all the subgroups of G of order 2. Which of them are normal?

ii) Now list all the remaining proper subgroups of G. [There are 6+3 of them.]

iii) For each proper normal subgroup N of G, describe the quotient group G/N.

iv) Show that G is not isomorphic to the alternating group A_4.

6D Algebra and Geometry

State the conditions on a matrix A that ensure it represents a rotation of $\mathbb{R}^3$ with respect to the standard basis.

Check that the matrix

$$A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$

represents a rotation. Find its axis of rotation $\mathbf{n}$.

Let Π be the plane perpendicular to the axis $\mathbf{n}$. The matrix A induces a rotation of Π by an angle θ. Find $\cos\theta$.

7D Algebra and Geometry

Let A be a real symmetric matrix. Show that all the eigenvalues of A are real, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Give an example of a non-zero *complex* (2×2) symmetric matrix whose only eigenvalues are zero. Is it diagonalisable?

8D Algebra and Geometry

Compute the characteristic polynomial of

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4-s & 2s-2 \\ 0 & -2s+2 & 4s-1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A for all values of s.

For which values of s is A diagonalisable?

9C Vector Calculus

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ state if the following implications are true or false. (No justification is required.)

(i) f is differentiable $\Rightarrow$ f is continuous.

(ii) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist $\Rightarrow$ f is continuous.

(iii) directional derivatives $\dfrac{\partial f}{\partial \mathbf{n}}$ exist for all unit vectors $\mathbf{n} \in \mathbb{R}^2 \Rightarrow f$ is differentiable.

(iv) f is differentiable $\Rightarrow \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous.

(v) all second order partial derivatives of f exist $\Rightarrow \dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$.

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that f is continuous at $(0,0)$ and find the partial derivatives $\dfrac{\partial f}{\partial x}(0,y)$ and $\dfrac{\partial f}{\partial y}(x,0)$. Then show that f is differentiable at $(0,0)$ and find its derivative. Investigate whether the second order partial derivatives $\dfrac{\partial^2 f}{\partial x\,\partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}(0,0)$ are the same. Are the second order partial derivatives of f at $(0,0)$ continuous? Justify your answer.

10C Vector Calculus

Explain what is meant by an exact differential. The three-dimensional vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = \left(e^x z^3 + 3x^2(e^y - e^z),\, e^y(x^3 - z^3),\, 3z^2(e^x - e^y) - e^z x^3\right).$$

Find the most general function that has $\mathbf{F} \cdot \mathbf{dx}$ as its differential.

Hence show that the line integral

$$\int_{P_1}^{P_2} \mathbf{F} \cdot \mathbf{dx}$$

along any path in $\mathbb{R}^3$ between points $P_1 = (0, a, 0)$ and $P_2 = (b, b, b)$ vanishes for any values of a and b.

The two-dimensional vector field $\mathbf{G}$ is defined at all points in $\mathbb{R}^2$ except $(0, 0)$ by

$$\mathbf{G} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right).$$

($\mathbf{G}$ is not defined at $(0, 0)$.) Show that

$$\oint_C \mathbf{G} \cdot \mathbf{dx} = 2\pi$$

for any closed curve C in $\mathbb{R}^2$ that goes around $(0, 0)$ anticlockwise precisely once without passing through $(0, 0)$.

11C Vector Calculus

Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0, 0, 0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, 0, 0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(\frac{-1}{4}, 0, 0)$. The eccentrically shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1. Is there a point in the interior of S_3 where a test particle would remain stably at rest? Justify your answer.

12C Vector Calculus

State (without proof) the divergence theorem for a vector field $\mathbf{F}$ with continuous first-order partial derivatives throughout a volume V enclosed by a bounded oriented piecewise-smooth non-self-intersecting surface S.

By calculating the relevant volume and surface integrals explicitly, verify the divergence theorem for the vector field

$$\mathbf{F} = \left(x^3 + 2xy^2,\, y^3 + 2yz^2,\, z^3 + 2zx^2\right),$$

defined within a sphere of radius R centred at the origin.

Suppose that functions ϕ, ψ are continuous and that their first and second partial derivatives are all also continuous in a region V bounded by a smooth surface S.

Show that

$$(1) \qquad \int_V (\phi \nabla^2 \psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS}\,.$$

$$(2) \qquad \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS} - \int_S \psi \boldsymbol{\nabla}\phi \cdot \mathbf{dS}\,.$$

Hence show that if $\rho(\mathbf{x})$ is a continuous function on V and $g(\mathbf{x})$ a continuous function on S and ϕ_1 and ϕ_2 are two continuous functions such that

$$\nabla^2 \phi_1(\mathbf{x}) = \nabla^2 \phi_2(\mathbf{x}) = \rho(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ in } V\text{, and}$$
$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ on } S,$$

then $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x}$ in V.