UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 28th May 2004 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **B** *and* **F** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

By writing $y(x) = mx$ where m is a constant, solve the differential equation

$$\frac{dy}{dx} = \frac{x - 2y}{2x + y}$$

and find the possible values of m.

Describe the isoclines of this differential equation and sketch the flow vectors. Use these to sketch at least two characteristically different solution curves.

Now, by making the substitution $y(x) = xu(x)$ or otherwise, find the solution of the differential equation which satisfies $y(0) = 1$.

2B Differential Equations

Find two linearly independent solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2y = 0\,.$$

Find also the solution of

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2y = e^{-px}$$

that satisfies

$$y = 0,\ \frac{dy}{dx} = 0 \quad \text{at } x = 0\,.$$

3F Probability

Define the covariance, $\operatorname{cov}(X, Y)$, of two random variables X and Y.

Prove, or give a counterexample to, each of the following statements.

(a) For any random variables X, Y, Z

$$\operatorname{cov}(X + Y, Z) = \operatorname{cov}(X, Z) + \operatorname{cov}(Y, Z).$$

(b) If X and Y are identically distributed, not necessarily independent, random variables then

$$\operatorname{cov}(X + Y, X - Y) = 0.$$

4F Probability

The random variable X has probability density function

$$f(x) = \begin{cases} cx(1-x) & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine c, and the mean and variance of X.

SECTION II

5B Differential Equations

Construct a series solution $y = y_1(x)$ valid in the neighbourhood of $x = 0$, for the differential equation

$$\frac{d^2y}{dx^2} + 4x^3\frac{dy}{dx} + x^2 y = 0\,,$$

satisfying

$$y_1 = 1,\ \frac{dy_1}{dx} = 0 \quad \text{at } x = 0\,.$$

Find also a second solution $y = y_2(x)$ which satisfies

$$y_2 = 0,\ \frac{dy_2}{dx} = 1 \quad \text{at } x = 0\,.$$

Obtain an expression for the Wronskian of these two solutions and show that

$$y_2(x) = y_1(x)\int_0^x \frac{e^{-\xi^4}}{y_1^2(\xi)}\,d\xi\,.$$

6B Differential Equations

Two solutions of the recurrence relation

$$x_{n+2} + b(n)x_{n+1} + c(n)x_n = 0$$

are given as p_n and q_n, and their Wronskian is defined to be

$$W_n = p_n q_{n+1} - p_{n+1} q_n\,.$$

Show that

$$W_{n+1} = W_1 \prod_{m=1}^{n} c(m)\,. \qquad (*)$$

Suppose that $b(n) = \alpha$, where α is a real constant lying in the range $[-2, 2]$, and that $c(n) = 1$. Show that two solutions are $x_n = e^{in\theta}$ and $x_n = e^{-in\theta}$, where $\cos\theta = -\alpha/2$. Evaluate the Wronskian of these two solutions and verify $(*)$.

7B Differential Equations

Show how a second-order differential equation $\ddot{x} = f(x, \dot{x})$ may be transformed into a pair of coupled first-order equations. Explain what is meant by a *critical point* on the phase diagram for a pair of first-order equations. Hence find the critical points of the following equations. Describe their stability type, sketching their behaviour near the critical points on a phase diagram.

(i) $\ddot{x} + \cos x = 0$

(ii) $\ddot{x} + x(x^2 + \lambda x + 1) = 0, \quad$ for $\lambda = 1,\ 5/2\,.$

Sketch the phase portraits of these equations marking clearly the direction of flow.

8B Differential Equations

Construct the general solution of the system of equations

$$\begin{aligned} \dot{x} + 4x + 3y &= 0 \\ \dot{y} + 4y - 3x &= 0 \end{aligned}$$

in the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{x} = \sum_{j=1}^{2} a_j \mathbf{x}^{(j)} e^{\lambda_j t}$$

and find the eigenvectors $\mathbf{x}^{(j)}$ and eigenvalues λ_j.

Explain what is meant by resonance in a forced system of linear differential equations.

Consider the forced system

$$\begin{aligned} \dot{x} + 4x + 3y &= \sum_{j=1}^{2} p_j e^{\lambda_j t} \\ \dot{y} + 4y - 3x &= \sum_{j=1}^{2} q_j e^{\lambda_j t}\,. \end{aligned}$$

Find conditions on p_j and q_j $(j = 1, 2)$ such that there is no resonant response to the forcing.

9F Probability

Let X be a positive-integer valued random variable. Define its *probability generating function* p_X. Show that if X and Y are independent positive-integer valued random variables, then $p_{X+Y} = p_X p_Y$.

A non-standard pair of dice is a pair of six-sided unbiased dice whose faces are numbered with strictly positive integers in a non-standard way (for example, $(2,2,2,3,5,7)$ and $(1,1,5,6,7,8)$). Show that there exists a non-standard pair of dice A and B such that when thrown

$$P\{\text{total shown by } A \text{ and } B \text{ is } n\} = P\{\text{total shown by pair of ordinary dice is } n\}$$

for all $2 \leqslant n \leqslant 12$.

[*Hint:* $(x+x^2+x^3+x^4+x^5+x^6) = x(1+x)(1+x^2+x^4) = x(1+x+x^2)(1+x^3)$.]

10F Probability

Define the *conditional probability* $P(A \mid B)$ of the event A given the event B.

A bag contains four coins, each of which when tossed is equally likely to land on either of its two faces. One of the coins shows a head on each of its two sides, while each of the other three coins shows a head on only one side. A coin is chosen at random, and tossed three times in succession. If heads turn up each time, what is the probability that if the coin is tossed once more it will turn up heads again? Describe the sample space you use and explain carefully your calculations.

11F Probability

The random variables X_1 and X_2 are independent, and each has an exponential distribution with parameter λ. Find the joint density function of

$$Y_1 = X_1 + X_2\,, \quad Y_2 = X_1/X_2\,,$$

and show that Y_1 and Y_2 are independent. What is the density of Y_2?

12F Probability

Let $A_1, A_2, \ldots, A_r$ be events such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Show that the number N of events that occur satisfies

$$P(N = 0) \quad = \quad 1 - \sum_{i=1}^{r} P(A_i)\,.$$

Planet Zog is a sphere with centre O. A number N of spaceships land at random on its surface, their positions being independent, each uniformly distributed over the surface. A spaceship at A is in direct radio contact with another point B on the surface if $\angle AOB < \frac{\pi}{2}$. Calculate the probability that every point on the surface of the planet is in direct radio contact with at least one of the N spaceships.

[*Hint*: The intersection of the surface of a sphere with a plane through the centre of the sphere is called a *great circle*. You may find it helpful to use the fact that N random great circles partition the surface of a sphere into $N(N-1)+2$ disjoint regions with probability one.]