MATHEMATICAL TRIPOS Part IA 2003

List of Courses

1/I/1B **Algebra and Geometry**

(a) Write the permutation

$$(123)(234)$$

as a product of disjoint cycles. Determine its order. Compute its sign.

(b) Elements x and y of a group G are *conjugate* if there exists a $g \in G$ such that $gxg^{-1} = y$.

Show that if permutations x and y are conjugate, then they have the same sign and the same order. Is the converse true? (Justify your answer with a proof or counter-example.)

1/I/2D **Algebra and Geometry**

Find the characteristic equation, the eigenvectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and the corresponding eigenvalues $\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}, \lambda_{\mathbf{d}}$ of the matrix

$$A = \begin{pmatrix} i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & i \end{pmatrix} .$$

Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ spans the complex vector space $\mathbb{C}^4$.

Consider the four subspaces of $\mathbb{C}^4$ defined parametrically by

$$\mathbf{z} = s\mathbf{a}, \quad \mathbf{z} = s\mathbf{b}, \quad \mathbf{z} = s\mathbf{c}, \quad \mathbf{z} = s\mathbf{d} \qquad (\mathbf{z} \in \mathbb{C}^4,\ s \in \mathbb{C}) .$$

Show that multiplication by A maps three of these subspaces onto themselves, and the remaining subspace into a smaller subspace to be specified.

1/II/5B **Algebra and Geometry**

(a) In the standard basis of $\mathbb{R}^2$, write down the matrix for a rotation through an angle θ about the origin.

(b) Let A be a real 3×3 matrix such that $\det A = 1$ and $AA^{\mathrm{T}} = I$, where A^{T} is the transpose of A.

(i) Suppose that A has an eigenvector $\mathbf{v}$ with eigenvalue 1. Show that A is a rotation through an angle θ about the line through the origin in the direction of $\mathbf{v}$, where $\cos\theta = \frac{1}{2}(\operatorname{trace} A - 1)$.

(ii) Show that A must have an eigenvector $\mathbf{v}$ with eigenvalue 1.

1/II/6A **Algebra and Geometry**

Let α be a linear map

$$\alpha : \mathbb{R}^3 \to \mathbb{R}^3 .$$

Define the kernel K and image I of α.

Let $\mathbf{y} \in \mathbb{R}^3$. Show that the equation $\alpha\mathbf{x} = \mathbf{y}$ has a solution $\mathbf{x} \in \mathbb{R}^3$ if and only if $\mathbf{y} \in I$.

Let α have the matrix

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t & -2b \\ 1 & t & 0 \end{pmatrix}$$

with respect to the standard basis, where $b \in \mathbb{R}$ and t is a real variable. Find K and I for α. Hence, or by evaluating the determinant, show that if $0 < b < 2$ and $\mathbf{y} \in I$ then the equation $\alpha\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x} \in \mathbb{R}^3$ for all values of t.

1/II/7B **Algebra and Geometry**

(i) State the orbit-stabilizer theorem for a group G acting on a set X.

(ii) Denote the group of *all* symmetries of the cube by G. Using the orbit-stabilizer theorem, show that G has 48 elements.

Does G have any non-trivial normal subgroups?

Let L denote the line between two diagonally opposite vertices of the cube, and let

$$H = \{g \in G \mid gL = L\}$$

be the subgroup of symmetries that preserve the line. Show that H is isomorphic to the direct product $S_3 \times C_2$, where S_3 is the symmetric group on 3 letters and C_2 is the cyclic group of order 2.

1/II/8D Algebra and Geometry

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy–Schwarz inequality, and deduce the triangle inequality.

By means of a sketch, give a geometric interpretation of the scalar product $\mathbf{x} \cdot \mathbf{y}$ in the case $n = 3$, relating the value of $\mathbf{x} \cdot \mathbf{y}$ to the angle α between the directions of $\mathbf{x}$ and $\mathbf{y}$.

What is a unit vector? Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u}\,.$$

Show that

(i) for any fixed, linearly independent $\mathbf{u}$ and $\mathbf{v}$, the minimum of S over $\mathbf{w}$ is attained when $\mathbf{w} = \lambda(\mathbf{u} + \mathbf{v})$ for some $\lambda \in \mathbb{R}$;

(ii) $\lambda \leqslant -\frac{1}{2}$ in all cases;

(iii) $\lambda = -1$ and $S = -3/2$ in the case where $\mathbf{u} \cdot \mathbf{v} = \cos(2\pi/3)$.

3/I/1A Algebra and Geometry

The mapping α of $\mathbb{R}^3$ into itself is a reflection in the plane $x_2 = x_3$. Find the matrix A of α with respect to any basis of your choice, which should be specified.

The mapping β of $\mathbb{R}^3$ into itself is a rotation about the line $x_1 = x_2 = x_3$ through $2\pi/3$, followed by a dilatation by a factor of 2. Find the matrix B of β with respect to a choice of basis that should again be specified.

Show explicitly that

$$B^3 = 8A^2$$

and explain why this must hold, irrespective of your choices of bases.

3/I/2B Algebra and Geometry

Show that if a group G contains a normal subgroup of order 3, and a normal subgroup of order 5, then G contains an element of order 15.

Give an example of a group of order 10 with no element of order 10.

3/II/5E Algebra and Geometry

(a) Show, using vector methods, that the distances from the centroid of a tetrahedron to the centres of opposite pairs of edges are equal. If the three distances are u, v, w and if a, b, c, d are the distances from the centroid to the vertices, show that

$$u^2 + v^2 + w^2 = \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2).$$

[*The centroid of k points in $\mathbb{R}^3$ with position vectors $\mathbf{x}_i$ is the point with position vector* $\dfrac{1}{k}\sum \mathbf{x}_i$.]

(b) Show that

$$|\mathbf{x} - \mathbf{a}|^2 \cos^2 \alpha = [(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n}]^2,$$

with $\mathbf{n}^2 = 1$, is the equation of a right circular double cone whose vertex has position vector $\mathbf{a}$, axis of symmetry $\mathbf{n}$ and opening angle α.

Two such double cones, with vertices $\mathbf{a}_1$ and $\mathbf{a}_2$, have parallel axes and the same opening angle. Show that if $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 \neq \mathbf{0}$, then the intersection of the cones lies in a plane with unit normal

$$\mathbf{N} = \frac{\mathbf{b}\cos^2\alpha - \mathbf{n}(\mathbf{n}\cdot\mathbf{b})}{\sqrt{\mathbf{b}^2\cos^4\alpha + (\mathbf{b}\cdot\mathbf{n})^2(1 - 2\cos^2\alpha)}}\,.$$

3/II/6E Algebra and Geometry

Derive an expression for the triple scalar product $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3$ in terms of the determinant of the matrix E whose rows are given by the components of the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Use the geometrical interpretation of the cross product to show that $\mathbf{e}_a$, $a = 1, 2, 3$, will be a *not necessarily orthogonal* basis for $\mathbb{R}^3$ as long as $\det E \neq 0$.

The rows of another matrix $\hat{E}$ are given by the components of three other vectors $\hat{\mathbf{e}}_b$, $b = 1, 2, 3$. By considering the matrix $E\hat{E}^{\mathrm{T}}$, where $^{\mathrm{T}}$ denotes the transpose, show that there is a unique choice of $\hat{E}$ such that $\hat{\mathbf{e}}_b$ is also a basis and

$$\mathbf{e}_a \cdot \hat{\mathbf{e}}_b = \delta_{ab}.$$

Show that the new basis is given by

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3} \qquad \text{etc.}$$

Show that if either $\mathbf{e}_a$ or $\hat{\mathbf{e}}_b$ is an orthonormal basis then E is a rotation matrix.

3/II/7B Algebra and Geometry

Let G be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $X = \{\alpha, \beta, \gamma\}$ be a set of three distinct points in $\mathbb{C} \cup \{\infty\}$.

(i) Show that there exists a $g \in G$ sending α to 0, β to 1, and γ to ∞.

(ii) Hence show that if $H = \{g \in G \mid gX = X\}$, then H is isomorphic to S_3, the symmetric group on 3 letters.

3/II/8B Algebra and Geometry

(a) Determine the characteristic polynomial and the eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

Is it diagonalizable?

(b) Show that an $n \times n$ matrix A with characteristic polynomial $f(t) = (t - \mu)^n$ is diagonalizable if and only if $A = \mu I$.

1/I/3B Analysis

Define what it means for a function of a real variable to be *differentiable* at $x \in \mathbb{R}$.

Prove that if a function is differentiable at $x \in \mathbb{R}$, then it is continuous there.

Show directly from the definition that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at 0 with derivative 0.

Show that the derivative $f'(x)$ is not continuous at 0.

1/I/4C Analysis

Explain what is meant by the *radius of convergence* of a power series.

Find the radius of convergence R of each of the following power series:

$$\text{(i)} \quad \sum_{n=1}^{\infty} n^{-2} z^n, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \left(n + \frac{1}{2^n} \right) z^n .$$

In each case, determine whether the series converges on the circle $|z| = R$.

1/II/9F Analysis

Prove the Axiom of Archimedes.

Let x be a real number in $[0, 1]$, and let m, n be positive integers. Show that the limit

$$\lim_{m \to \infty} \left[\lim_{n \to \infty} \cos^{2n} (m! \pi x) \right]$$

exists, and that its value depends on whether x is rational or irrational.

[*You may assume standard properties of the cosine function provided they are clearly stated.*]

1/II/10F **Analysis**

State without proof the *Integral Comparison Test* for the convergence of a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms.

Determine for which positive real numbers α the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges.

In each of the following cases determine whether the series is convergent or divergent:

(i) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n}$,

(ii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n \log n)\ (\log\log n)^2}$,

(iii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{(1+1/n)} \log n}$.

1/II/11B **Analysis**

Let $f : [a, b] \to \mathbb{R}$ be continuous. Define the *integral* $\int_a^b f(x)dx$. (You are not asked to prove existence.)

Suppose that m, M are real numbers such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. Stating clearly any properties of the integral that you require, show that

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)\,.$$

The function $g : [a, b] \to \mathbb{R}$ is continuous and non-negative. Show that

$$m \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M \int_a^b g(x)dx\,.$$

Now let f be continuous on $[0, 1]$. By suitable choice of g show that

$$\lim_{n\to\infty} \int_0^{1/\sqrt{n}} nf(x)e^{-nx}dx = f(0)\ ,$$

and by making an appropriate change of variable, or otherwise, show that

$$\lim_{n\to\infty} \int_0^1 nf(x)e^{-nx}dx = f(0)\,.$$

1/II/12C **Analysis**

State carefully the formula for integration by parts for functions of a real variable.

Let $f : (-1,1) \to \mathbb{R}$ be infinitely differentiable. Prove that for all $n \geqslant 1$ and all $t \in (-1,1)$,

$$f(t) = f(0) + f'(0)t + \frac{1}{2!}f''(0)t^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(0)t^{n-1} + \frac{1}{(n-1)!}\int_0^t f^{(n)}(x)(t-x)^{n-1}\,dx.$$

By considering the function $f(x) = \log(1-x)$ at $x = 1/2$, or otherwise, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

converges to $\log 2$.

2/I/1D **Differential Equations**

Consider the equation

$$\frac{dy}{dx} = 1 - y^2 \, . \qquad (*)$$

Using small line segments, sketch the flow directions in $x \geqslant 0$, $-2 \leqslant y \leqslant 2$ implied by the right-hand side of $(*)$. Find the general solution

(i) in $|y| < 1$,

(ii) in $|y| > 1$.

Sketch a solution curve in each of the three regions $y > 1$, $|y| < 1$, and $y < -1$.

2/I/2D **Differential Equations**

Consider the differential equation

$$\frac{dx}{dt} + Kx = 0 \, ,$$

where K is a positive constant. By using the approximate finite-difference formula

$$\frac{dx_n}{dt} = \frac{x_{n+1} - x_{n-1}}{2\delta t} \, ,$$

where δt is a positive constant, and where x_n denotes the function $x(t)$ evaluated at $t = n\delta t$ for integer n, convert the differential equation to a difference equation for x_n.

Solve both the differential equation and the difference equation for general initial conditions. Identify those solutions of the difference equation that agree with solutions of the differential equation over a finite interval $0 \leqslant t \leqslant T$ in the limit $\delta t \to 0$, and demonstrate the agreement. Demonstrate that the remaining solutions of the difference equation cannot agree with the solution of the differential equation in the same limit.

[*You may use the fact that, for bounded* $|u|$, $\lim_{\epsilon \to 0} (1 + \epsilon u)^{1/\epsilon} = e^u$.]

2/II/5D **Differential Equations**

(a) Show that if $\mu(x,y)$ is an integrating factor for an equation of the form

$$f(x,y)\,dy \;+\; g(x,y)\,dx \;=\; 0$$

then $\partial(\mu f)/\partial x \;=\; \partial(\mu g)/\partial y\,.$

Consider the equation

$$\cot x\; dy \;-\; \tan y\; dx \;=\; 0$$

in the domain $0 \leqslant x \leqslant \frac{1}{2}\pi\,,\;\; 0 \leqslant y \leqslant \frac{1}{2}\pi\,.$ Using small line segments, sketch the flow directions in that domain. Show that $\sin x\,\cos y$ is an integrating factor for the equation. Find the general solution of the equation, and sketch the family of solutions that occupies the larger domain $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi\,,\;\; -\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi\,.$

(b) The following example illustrates that the concept of integrating factor extends to higher-order equations. Multiply the equation

$$\left[y\,\frac{d^2y}{dx^2} \;+\; \left(\frac{dy}{dx}\right)^2 \right] \cos^2 x \;=\; 1$$

by $\sec^2 x$, and show that the result takes the form $\dfrac{d}{dx}h(x,y) = 0$, for some function $h(x,y)$ to be determined. Find a particular solution $y = y(x)$ such that $y(0) = 0$ with dy/dx finite at $x = 0\,,$ and sketch its graph in $0 \leqslant x < \frac{1}{2}\pi\,.$

2/II/6D **Differential Equations**

Define the *Wronskian* $W(x)$ associated with solutions of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y \;=\; 0$$

and show that

$$W(x) \;\propto\; \exp\left(-\int^x p(\xi)\,d\xi\right) \;.$$

Evaluate the expression on the right when $p(x) = -2/x$.

Given that $p(x) = -2/x$ and that $q(x) = -1$, show that solutions in the form of power series,

$$y \;=\; x^\lambda \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0)\,,$$

can be found if and only if $\lambda = 0$ or 3. By constructing and solving the appropriate recurrence relations, find the coefficients a_n for each power series.

You may assume that the equation is satisfied by $y = \cosh x - x\sinh x$ and by $y = \sinh x - x\cosh x$. Verify that these two solutions agree with the two power series found previously, and that they give the $W(x)$ found previously, up to multiplicative constants.

[*Hint:* $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\,,\;\; \sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\,.$]

2/II/7D **Differential Equations**

Consider the linear system

$$\dot{\mathbf{x}}(t) - A\mathbf{x}(t) = \mathbf{z}(t)$$

where the n-vector $\mathbf{z}(t)$ and the $n \times n$ matrix A are given; A has constant real entries, and has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and n linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$. Find the complementary function. Given a particular integral $\mathbf{x}_{\mathrm{p}}(t)$, write down the general solution. In the case $n = 2$ show that the complementary function is purely oscillatory, with no growth or decay, if and only if

$$\operatorname{trace} A = 0 \qquad \text{and} \qquad \det A > 0 \; .$$

Consider the same case $n = 2$ with $\operatorname{trace} A = 0$ and $\det A > 0$ and with

$$\mathbf{z}(t) = \mathbf{a}_1 \exp(i\omega_1 t) + \mathbf{a}_2 \exp(i\omega_2 t) \, ,$$

where $\omega_1, \ \omega_2$ are given real constants. Find a particular integral when

(i) $i\omega_1 \neq \lambda_1$ and $i\omega_2 \neq \lambda_2$;

(ii) $i\omega_1 \neq \lambda_1$ but $i\omega_2 = \lambda_2$.

In the case

$$A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$$

with $\mathbf{z}(t) = \begin{pmatrix} 2 \\ 3i-1 \end{pmatrix} \exp(3it)$, find the solution subject to the initial condition $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$.

2/II/8D **Differential Equations**

For all solutions of

$$\begin{aligned} \dot{x} &= \tfrac{1}{2}\alpha x + y - 2y^3, \\ \dot{y} &= -x \end{aligned}$$

show that $dK/dt = \alpha x^2$ where

$$K = K(x,y) = x^2 + y^2 - y^4 \; .$$

In the case $\alpha = 0$, analyse the properties of the critical points and sketch the phase portrait, including the special contours for which $K(x,y) = \frac{1}{4}$. Comment on the asymptotic behaviour, as $t \to \infty$, of solution trajectories that pass near each critical point, indicating whether or not any such solution trajectories approach from, or recede to, infinity.

Briefly discuss how the picture changes when α is made small and positive, using your result for dK/dt to describe, in qualitative terms, how solution trajectories cross K-contours.

4/I/3E **Dynamics**

Because of an accident on launching, a rocket of unladen mass M lies horizontally on the ground. It initially contains fuel of mass m_0, which ignites and is emitted horizontally at a constant rate and at uniform speed u relative to the rocket. The rocket is initially at rest. If the coefficient of friction between the rocket and the ground is μ, and the fuel is completely burnt in a total time T, show that the final speed of the rocket is

$$u \log\left(\frac{M+m_0}{M}\right) - \mu g T.$$

4/I/4E **Dynamics**

Write down an expression for the total momentum $\mathbf{P}$ and angular momentum $\mathbf{L}$ with respect to an origin O of a system of n point particles of masses m_i, position vectors (with respect to O) $\mathbf{x}_i$, and velocities $\mathbf{v}_i$, $i = 1, \ldots, n$.

Show that with respect to a new origin O' the total momentum $\mathbf{P}'$ and total angular momentum $\mathbf{L}'$ are given by

$$\mathbf{P}' = \mathbf{P}, \qquad \mathbf{L}' = \mathbf{L} - \mathbf{b} \times \mathbf{P},$$

and hence

$$\mathbf{L}' \cdot \mathbf{P}' = \mathbf{L} \cdot \mathbf{P},$$

where $\mathbf{b}$ is the constant vector displacement of O' with respect to O. How does $\mathbf{L} \times \mathbf{P}$ change under change of origin?

Hence show that **either**

(1) the total momentum vanishes and the total angular momentum is independent of origin, **or**

(2) by choosing $\mathbf{b}$ in a way that should be specified, the total angular momentum with respect to O' can be made parallel to the total momentum.

4/II/9E **Dynamics**

Write down the equation of motion for a point particle with mass m, charge e, and position vector $\mathbf{x}(t)$ moving in a time-dependent magnetic field $\mathbf{B}(\mathbf{x},t)$ with vanishing electric field, and show that the kinetic energy of the particle is constant. If the magnetic field is constant in direction, show that the component of velocity in the direction of $\mathbf{B}$ is constant. Show that, in general, the angular momentum of the particle is not conserved.

Suppose that the magnetic field is independent of time and space and takes the form $\mathbf{B} = (0,0,B)$ and that $\dot{A}$ is the rate of change of area swept out by a radius vector joining the origin to the projection of the particle's path on the (x,y) plane. Obtain the equation

$$\frac{d}{dt}\left(m\dot{A} + \frac{eBr^2}{4}\right) = 0\ , \qquad (*)$$

where (r,θ) are plane polar coordinates. Hence obtain an equation replacing the equation of conservation of angular momentum.

Show further, using energy conservation and $(*)$, that the equations of motion in plane polar coordinates may be reduced to the first order non-linear system

$$\dot{r} = \sqrt{v^2 - \left(\frac{2c}{mr} - \frac{erB}{2m}\right)^2},$$

$$\dot{\theta} = \frac{2c}{mr^2} - \frac{eB}{2m}\ ,$$

where v and c are constants.

4/II/10E **Dynamics**

Write down the equations of motion for a system of n gravitating particles with masses m_i, and position vectors $\mathbf{x}_i$, $i = 1, 2, \ldots, n$.

The particles undergo a motion for which $\mathbf{x}_i(t) = a(t)\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Show that the equations of motion will be satisfied as long as the function $a(t)$ satisfies

$$\ddot{a} = -\frac{\Lambda}{a^2} , \qquad (*)$$

where Λ is a constant and the vectors $\mathbf{a}_i$ satisfy

$$\Lambda m_i \mathbf{a}_i = \mathbf{G}_i = \sum_{j \neq i} \frac{G m_i m_j \left(\mathbf{a}_i - \mathbf{a}_j\right)}{|\mathbf{a}_i - \mathbf{a}_j|^3} . \qquad (**)$$

Show that $(*)$ has as first integral

$$\frac{\dot{a}^2}{2} - \frac{\Lambda}{a} = \frac{k}{2} ,$$

where k is another constant. Show that

$$\mathbf{G}_i = \boldsymbol{\nabla}_i W ,$$

where $\boldsymbol{\nabla}_i$ is the gradient operator with respect to $\mathbf{a}_i$ and

$$W = -\sum_i \sum_{j<i} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|} .$$

Using Euler's theorem for homogeneous functions (see below), or otherwise, deduce that

$$\sum_i \mathbf{a}_i \cdot \mathbf{G}_i = -W .$$

Hence show that all solutions of $(**)$ satisfy

$$\Lambda I = -W$$

where

$$I = \sum_i m_i \mathbf{a}_i^2 .$$

Deduce that Λ must be positive and that the total kinetic energy plus potential energy of the system of particles is equal to $\frac{k}{2} I$.

[*Euler's theorem states that if*

$$f(\lambda x, \lambda y, \lambda z, \ldots) = \lambda^p f(x, y, z, \ldots) ,$$

then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} + \ldots = p f .]$$

4/II/11E **Dynamics**

State the parallel axis theorem and use it to calculate the moment of inertia of a uniform hemisphere of mass m and radius a about an axis through its centre of mass and parallel to the base.

[*You may assume that the centre of mass is located at a distance* $\frac{3}{8}a$ *from the flat face of the hemisphere, and that the moment of inertia of a full sphere about its centre is* $\frac{2}{5}Ma^2$, *with* $M = 2m$.]

The hemisphere initially rests on a rough horizontal plane with its base vertical. It is then released from rest and subsequently rolls on the plane without slipping. Let θ be the angle that the base makes with the horizontal at time t. Express the instantaneous speed of the centre of mass in terms of b and the rate of change of θ, where b is the instantaneous distance from the centre of mass to the point of contact with the plane. Hence write down expressions for the kinetic energy and potential energy of the hemisphere and deduce that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{(28 - 15\cos\theta)a}\,.$$

4/II/12E **Dynamics**

Let (r, θ) be plane polar coordinates and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ unit vectors in the direction of increasing r and θ respectively. Show that the velocity of a particle moving in the plane with polar coordinates $\big(r(t),\ \theta(t)\big)$ is given by

$$\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\,,$$

and that the unit normal $\mathbf{n}$ to the particle path is parallel to

$$r\dot{\theta}\mathbf{e}_r - \dot{r}\mathbf{e}_\theta\,.$$

Deduce that the perpendicular distance p from the origin to the tangent of the curve $r = r(\theta)$ is given by

$$\frac{r^2}{p^2} = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2.$$

The particle, whose mass is m, moves under the influence of a central force with potential $V(r)$. Use the conservation of energy E and angular momentum h to obtain the equation

$$\frac{1}{p^2} = \frac{2m\big(E - V(r)\big)}{h^2}\,.$$

Hence express θ as a function of r as the integral

$$\theta = \int \frac{hr^{-2}dr}{\sqrt{2m\big(E - V_{\text{eff}}(r)\big)}}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{h^2}{2mr^2}\,.$$

Evaluate the integral and describe the orbit when $V(r) = \dfrac{c}{r^2}$, with c a positive constant.

4/I/1C **Numbers and Sets**

(i) Prove by induction or otherwise that for every $n \geqslant 1$,

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2 .$$

(ii) Show that the sum of the first n positive cubes is divisible by 4 if and only if $n \equiv 0$ or $3 \pmod 4$.

4/I/2C **Numbers and Sets**

What is an *equivalence relation*? For each of the following pairs $(X, \sim)$, determine whether or not $\sim$ is an equivalence relation on X:

(i) $X = \mathbb{R}$, $x \sim y$ iff $x - y$ is an even integer;

(ii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{R}$;

(iii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{Z}$;

(iv) $X = \mathbb{Z} \setminus \{0\}$, $x \sim y$ iff $x^2 - y^2$ is ± 1 times a perfect square.

4/II/5C **Numbers and Sets**

Define what is meant by the term *countable*. Show directly from your definition that if X is countable, then so is any subset of X.

Show that $\mathbb{N} \times \mathbb{N}$ is countable. Hence or otherwise, show that a countable union of countable sets is countable. Show also that for any $n \geqslant 1$, $\mathbb{N}^n$ is countable.

A function $f : \mathbb{Z} \to \mathbb{N}$ is *periodic* if there exists a positive integer m such that, for every $x \in \mathbb{Z}$, $f(x + m) = f(x)$. Show that the set of periodic functions $f : \mathbb{Z} \to \mathbb{N}$ is countable.

4/II/6C Numbers and Sets

(i) Prove Wilson's theorem: if p is prime then $(p-1)! \equiv -1 \pmod p$.

Deduce that if $p \equiv 1 \pmod 4$ then

$$\left(\left(\frac{p-1}{2}\right)!\right)^2 \equiv -1 \pmod p.$$

(ii) Suppose that p is a prime of the form $4k+3$. Show that if $x^4 \equiv 1 \pmod p$ then $x^2 \equiv 1 \pmod p$.

(iii) Deduce that if p is an odd prime, then the congruence

$$x^2 \equiv -1 \pmod p$$

has exactly two solutions (modulo p) if $p \equiv 1 \pmod 4$, and none otherwise.

4/II/7C Numbers and Sets

Let m, n be integers. Explain what is their *greatest common divisor* (m,n). Show from your definition that, for any integer k, $(m,n) = (m+kn,n)$.

State Bezout's theorem, and use it to show that if p is prime and p divides mn, then p divides at least one of m and n.

The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ... is defined by $x_0 = 0$, $x_1 = 1$ and $x_{n+1} = x_n + x_{n-1}$ for $n \geqslant 1$. Prove:

(i) $(x_{n+1}, x_n) = 1$ and $(x_{n+2}, x_n) = 1$ for every $n \geqslant 0$;

(ii) $x_{n+3} \equiv x_n \pmod 2$ and $x_{n+8} \equiv x_n \pmod 3$ for every $n \geqslant 0$;

(iii) if $n \equiv 0 \pmod 5$ then $x_n \equiv 0 \pmod 5$.

4/II/8C Numbers and Sets

Let X be a finite set with n elements. How many functions are there from X to X? How many relations are there on X?

Show that the number of relations R on X such that, for each $y \in X$, there exists at least one $x \in X$ with xRy, is $(2^n - 1)^n$.

Using the inclusion–exclusion principle or otherwise, deduce that the number of such relations R for which, in addition, for each $x \in X$, there exists at least one $y \in X$ with xRy, is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^n.$$

2/I/3F Probability

(a) Define the *probability generating function* of a random variable. Calculate the probability generating function of a binomial random variable with parameters n and p, and use it to find the mean and variance of the random variable.

(b) X is a binomial random variable with parameters n and p, Y is a binomial random variable with parameters m and p, and X and Y are independent. Find the distribution of $X+Y$; that is, determine $P\{X+Y=k\}$ for all possible values of k.

2/I/4F Probability

The random variable X is uniformly distributed on the interval $[0,1]$. Find the distribution function and the probability density function of Y, where

$$Y = \frac{3X}{1-X}\,.$$

2/II/9F Probability

State the inclusion-exclusion formula for the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs.

After a party the n guests take coats randomly from a pile of their n coats. Calculate the probability that no-one goes home with the correct coat.

Let $p(m,n)$ be the probability that exactly m guests go home with the correct coats. By relating $p(m,n)$ to $p(0,n-m)$, or otherwise, determine $p(m,n)$ and deduce that

$$\lim_{n\to\infty} p(m,n) = \frac{1}{em!}\,.$$

2/II/10F Probability

The random variables X and Y each take values in $\{0,1\}$, and their joint distribution $p(x,y) = P\{X=x,\ Y=y\}$ is given by

$$p(0,0)=a, \qquad p(0,1)=b, \qquad p(1,0)=c, \qquad p(1,1)=d\,.$$

Find necessary and sufficient conditions for X and Y to be

(i) uncorrelated;

(ii) independent.

Are the conditions established in (i) and (ii) equivalent?

2/II/11F Probability

A laboratory keeps a population of aphids. The probability of an aphid passing a day uneventfully is $q < 1$. Given that a day is not uneventful, there is probability r that the aphid will have one offspring, probability s that it will have two offspring and probability t that it will die, where $r + s + t = 1$. Offspring are ready to reproduce the next day. The fates of different aphids are independent, as are the events of different days. The laboratory starts out with one aphid.

Let X_1 be the number of aphids at the end of the first day. What is the expected value of X_1? Determine an expression for the probability generating function of X_1.

Show that the probability of extinction does not depend on q, and that if $2r+3s \leqslant 1$ then the aphids will certainly die out. Find the probability of extinction if $r = 1/5,\ s = 2/5$ and $t = 2/5$.

[*Standard results on branching processes may be used without proof, provided that they are clearly stated.*]

2/II/12F Probability

Planet Zog is a ball with centre O. Three spaceships A, B and C land at random on its surface, their positions being independent and each uniformly distributed on its surface. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Spaceships A and B can communicate directly by radio if $\angle AOB < \pi/2$, and similarly for spaceships B and C and spaceships A and C. Given angle $\angle AOB = \gamma < \pi/2$, calculate the probability that C can communicate directly with *either* A or B. Given angle $\angle AOB = \gamma > \pi/2$, calculate the probability that C can communicate directly with *both* A and B. Hence, or otherwise, show that the probability that all three spaceships can keep in in touch (with, for example, A communicating with B via C if necessary) is $(\pi + 2)/(4\pi)$.

3/I/3A **Vector Calculus**

Sketch the curve $y^2 = x^2 + 1$. By finding a parametric representation, or otherwise, determine the points on the curve where the radius of curvature is least, and compute its value there.

[*Hint: you may use the fact that the radius of curvature of a parametrized curve* $(x(t), y(t))$ *is* $(\dot{x}^2 + \dot{y}^2)^{3/2}/|\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.]

3/I/4A **Vector Calculus**

Suppose V is a region in $\mathbb{R}^3$, bounded by a piecewise smooth closed surface S, and $\phi(\mathbf{x})$ is a scalar field satisfying

$$\begin{aligned} \boldsymbol{\nabla}^2\phi &= 0 \quad \text{in } V, \\ \text{and} \quad \phi &= f(\mathbf{x}) \quad \text{on } S\,. \end{aligned}$$

Prove that ϕ is determined uniquely in V.

How does the situation change if the normal derivative of ϕ rather than ϕ itself is specified on S?

3/II/9A **Vector Calculus**

Let C be the closed curve that is the boundary of the triangle T with vertices at the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Specify a direction along C and consider the integral

$$\oint_C \mathbf{A} \cdot d\mathbf{x}\ ,$$

where $\mathbf{A} = (z^2 - y^2,\ x^2 - z^2,\ y^2 - x^2)$. Explain why the contribution to the integral is the same from each edge of C, and evaluate the integral.

State Stokes's theorem and use it to evaluate the surface integral

$$\int_T (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S}\ ,$$

the components of the normal to T being positive.

Show that $d\mathbf{S}$ in the above surface integral can be written in the form $(1,1,1)\,dy\,dz$. Use this to verify your result by a direct calculation of the surface integral.

3/II/10A Vector Calculus

Write down an expression for the Jacobian J of a transformation

$$(x, y, z) \to (u, v, w)\,.$$

Use it to show that

$$\int_D f \; dx\,dy\,dz = \int_\Delta \phi \;|J|\; du\,dv\,dw$$

where D is mapped one-to-one onto Δ, and

$$\phi(u, v, w) \;=\; f\big(x(u, v, w),\; y(u, v, w),\; z(u, v, w)\big)\,.$$

Find a transformation that maps the ellipsoid D,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\,,$$

onto a sphere. Hence evaluate

$$\int_D x^2\,dx\,dy\,dz\,.$$

3/II/11A Vector Calculus

(a) Prove the identity

$$\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G}) \;=\; (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G} \;+\; (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} \;+\; \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G}) \;+\; \mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F})\,.$$

(b) If $\mathbf{E}$ is an irrotational vector field ($\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E} = -\boldsymbol{\nabla}\phi$.

Show that

$$(2xy^2ze^{-x^2z},\; -2ye^{-x^2z},\; x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

3/II/12A **Vector Calculus**

State the divergence theorem. By applying this to $f(\mathbf{x})\mathbf{k}$, where $f(\mathbf{x})$ is a scalar field in a closed region V in $\mathbb{R}^3$ bounded by a piecewise smooth surface S, and $\mathbf{k}$ an arbitrary constant vector, show that

$$\int_V \nabla f \, dV = \int_S f \, d\mathbf{S} \,. \qquad (*)$$

A vector field $\mathbf{G}$ satisfies

$$\nabla \cdot \mathbf{G} = \rho(\mathbf{x})$$

$$\text{with} \quad \rho(\mathbf{x}) = \begin{cases} \rho_0 & |\mathbf{x}| \leqslant a \\ 0 & |\mathbf{x}| > a \,. \end{cases}$$

By applying the divergence theorem to $\int_V \nabla \cdot \mathbf{G} \, dV$, prove Gauss's law

$$\int_S \mathbf{G} \cdot d\mathbf{S} = \int_V \rho(\mathbf{x}) \, dV,$$

where S is the piecewise smooth surface bounding the volume V.

Consider the spherically symmetric solution

$$\mathbf{G}(\mathbf{x}) = G(r) \, \frac{\mathbf{x}}{r} \,,$$

where $r = |\mathbf{x}|$. By using Gauss's law with S a sphere of radius r, centre $\mathbf{0}$, in the two cases $0 < r \leqslant a$ and $r > a$, show that

$$\mathbf{G}(\mathbf{x}) = \begin{cases} \dfrac{\rho_0}{3}\,\mathbf{x} & r \leqslant a \\ \dfrac{\rho_0}{3}\left(\dfrac{a}{r}\right)^3 \mathbf{x} & r > a \,. \end{cases}$$

The scalar field $f(\mathbf{x})$ satisfies $\mathbf{G} = \nabla f$. Assuming that $f \to 0$ as $r \to \infty$, and that f is continuous at $r = a$, find f everywhere.

By using a symmetry argument, explain why $(*)$ is clearly satisfied for this f if S is any sphere centred at the origin.