UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Thursday 5 June 2003 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **3A, 22A** *should be in one bundle and* **1J, 14J** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

(i) Consider the continuous-time Markov chain $(X_t)_{t \geqslant 0}$ with state-space $\{1, 2, 3, 4\}$ and Q-matrix

$$Q = \begin{pmatrix} -2 & 0 & 0 & 2 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 5 & 2 & -8 \end{pmatrix}.$$

Set

$$Y_t = \begin{cases} X_t & \text{if } \; X_t \in \{1, 2, 3\} \\ 2 & \text{if } \; X_t = 4 \end{cases}$$

and

$$Z_t = \begin{cases} X_t & \text{if } \; X_t \in \{1, 2, 3\} \\ 1 & \text{if } \; X_t = 4. \end{cases}$$

Determine which, if any, of the processes $(Y_t)_{t \geqslant 0}$ and $(Z_t)_{t \geqslant 0}$ are Markov chains.

(ii) Find an invariant distribution for the chain $(X_t)_{t \geqslant 0}$ given in Part (i). Suppose $X_0 = 1$. Find, for all $t \geqslant 0$, the probability that $X_t = 1$.

2G Functional Analysis

(i) Let p be a point of the compact interval $I = [a, b] \subset \mathbb{R}$ and let $\delta_p : C(I) \to \mathbb{R}$ be defined by $\delta_p(f) = f(p)$. Show that

$$\delta_p : (C(I), || \cdot ||_\infty) \to \mathbb{R}$$

is a continuous, linear map but that

$$\delta_p : (C(I), || \cdot ||_1) \to \mathbb{R}$$

is not continuous.

(ii) Consider the space $C^{(n)}(I)$ of n-times continuously differentiable functions on the interval I. Write

$$||f||_\infty^{(n)} = \sum_{k=0}^{n} ||f^{(k)}||_\infty \quad \text{and} \quad ||f||_1^{(n)} = \sum_{r=0}^{n} ||f^{(k)}||_1$$

for $f \in C^{(n)}(I)$. Show that $(C^{(n)}(I), || \cdot ||_\infty^{(n)})$ is a complete normed space. Is the space $(C^{(n)}(I), || \cdot ||_1^{(n)})$ also complete?

Let $f : I \to I$ be an n-times continuously differentiable map and define

$$\mu_f : C^{(n)}(I) \to C^{(n)}(I) \quad \text{by} \quad g \mapsto g \circ f\,.$$

Show that μ_f is a continuous linear map when $C^{(n)}(I)$ is equipped with the norm $|| \cdot ||_\infty^{(n)}$.

3A Electromagnetism

(i) Given the electric field (in cartesian components)

$$\mathbf{E}(\mathbf{r},t) = \big(0, x/t^2, 0\big)\,,$$

use the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

to find $\mathbf{B}$ subject to the boundary condition that $|\mathbf{B}| \to 0$ as $t \to \infty$.

Let S be the planar rectangular surface in the xy-plane with corners at

$$(0,0,0)\,, \qquad (L,0,0)\,, \qquad (L,a,0)\,, \qquad (0,a,0)$$

where a is a constant and $L = L(t)$ is some function of time. The magnetic flux through S is given by the surface integral

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}.$$

Compute Φ as a function of t.

Let $\mathcal{C}$ be the closed rectangular curve that bounds the surface S, taken anticlockwise in the xy-plane, and let $\mathbf{v}$ be its velocity (which depends, in this case, on the segment of $\mathcal{C}$ being considered). Compute the line integral

$$\oint_{\mathcal{C}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}.$$

Hence verify that

$$\oint_{\mathcal{C}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = -\frac{d\Phi}{dt}\,. \tag{2}$$

(ii) A surface S is bounded by a time-dependent closed curve $\mathcal{C}(t)$ such that in time δt it sweeps out a volume δV. By considering the volume integral

$$\int_{\delta V} \nabla \cdot \mathbf{B}\, d\tau\,,$$

and using the divergence theorem, show that the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ implies that

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_{\mathcal{C}} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$$

where Φ is the magnetic flux through S as given in Part (i). Hence show, using (1) and Stokes' theorem, that (2) is a consequence of Maxwell's equations.

4D Dynamics of Differential Equations

(i) Define the Poincaré index of a curve $\mathcal{C}$ for a vector field $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$. Explain why the index is uniquely given by the sum of the indices for small curves around each fixed point within $\mathcal{C}$. Write down the indices for a saddle point and for a focus (spiral) or node, and show that the index of a periodic solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has index unity.

A particular system has a periodic orbit containing five fixed points, and two further periodic orbits. Sketch the possible arrangements of these orbits, assuming there are no degeneracies.

(ii) A dynamical system in $\mathbb{R}^2$ depending on a parameter μ has a homoclinic orbit when $\mu = \mu_0$. Explain how to determine the stability of this orbit, and sketch the different behaviours for $\mu < \mu_0$ and $\mu > \mu_0$ in the case that the orbit is stable.

Now consider the system

$$\dot{x} = y \ , \quad \dot{y} = x - x^2 + y(\alpha + \beta x)$$

where α, β are constants. Show that the origin is a saddle point, and that if there is an orbit homoclinic to the origin then α, β are related by

$$\oint y^2(\alpha + \beta x) dt = 0$$

where the integral is taken round the orbit. Evaluate this integral for small α, β by approximating y by its form when $\alpha = \beta = 0$. Hence give conditions on (small) α, β that lead to a stable homoclinic orbit at the origin. [*Note that* $y dt = dx$*.*]

5F Representation Theory

If $\rho_1 : G_1 \to GL(V_1)$ and $\rho_2 : G_2 \to GL(V_2)$ are representations of the finite groups G_1 and G_2 respectively, define the tensor product $\rho_1 \otimes \rho_2$ as a representation of the group $G_1 \times G_2$ and show that its character is given by

$$\chi_{\rho_1 \otimes \rho_2}(g_1, g_2) = \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2) \, .$$

Prove that

(a) if ρ_1 and ρ_2 are irreducible, then $\rho_1 \otimes \rho_2$ is an irreducible representation of $G_1 \times G_2$;

(b) each irreducible representation of $G_1 \times G_2$ is equivalent to a representation $\rho_1 \otimes \rho_2$ where each ρ_i is irreducible $(i = 1, 2)$.

Is every representation of $G_1 \times G_2$ the tensor product of a representation of G_1 and a representation of G_2?

6F Galois Theory

Let f be a separable polynomial of degree $n \geqslant 1$ over a field K. Explain what is meant by the Galois group $\mathrm{Gal}(f/K)$ of f over K. Explain how $\mathrm{Gal}(f/K)$ can be identified with a subgroup of the symmetric group S_n. Show that as a permutation group, $\mathrm{Gal}(f/K)$ is transitive if and only if f is irreducible over K.

Show that the Galois group of $f(X) = X^5 + 20X^2 - 2$ over $\mathbb{Q}$ is S_5, stating clearly any general results you use.

Now let $K/\mathbb{Q}$ be a finite extension of prime degree $p > 5$. By considering the degrees of the splitting fields of f over K and $\mathbb{Q}$, show that $\mathrm{Gal}(f/K) = S_5$ also.

7G Algebraic Topology

Define a covering map. Prove that any covering map induces an injective homomorphisms of fundamental groups.

Show that there is a non-trivial covering map of the real projective plane. Explain how to use this to find the fundamental group of the real projective plane.

8H Hilbert Spaces

Let $\mathcal{H}$ be the space of all functions on the real line $\mathbb{R}$ of the form $p(x)e^{-x^2/2}$, where p is a polynomial with complex coefficients. Make $\mathcal{H}$ into an inner-product space, in the usual way, by defining the inner product to be

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt, \quad f, g \in \mathcal{H}.$$

You should assume, without proof, that this equation does define an inner product on $\mathcal{H}$. Define the norm by $\|f\|_2 = \langle f, f\rangle^{1/2}$ for $f \in \mathcal{H}$. Now define a sequence of functions $(F_n)_{n\geqslant 0}$ on $\mathbb{R}$ by

$$F_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} .$$

Prove that (F_n) is an orthogonal sequence in $\mathcal{H}$ and that it spans $\mathcal{H}$.

For every $f \in \mathcal{H}$ define the Fourier transform $\widehat{f}$ of f by

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx, \quad t \in \mathbb{R}.$$

Show that

(a) $\widehat{F_n} = (-i)^n F_n$ for $n = 0, 1, 2, \ldots$;

(b) for all $f \in \mathcal{H}$ and $x \in \mathbb{R}$,

$$\widehat{\widehat{f}}(x) = f(-x)\,;$$

(c) $\|\widehat{f}\|_2 = \|f\|_2$ for all $f \in \mathcal{H}$.

9G Riemann Surfaces

Let L be the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for two non-zero complex numbers ω_1, ω_2 whose ratio is not real. Recall that the *Weierstrass function* $\wp$ is given by the series

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega \in L - \{0\}} \left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \right) ;$$

the function ζ is the (unique) odd anti-derivative of $-\wp$; and σ is defined by the conditions

$$\sigma'(u) = \zeta(u)\sigma(u) \qquad \text{and} \qquad \sigma'(0) = 1 \ .$$

(a) By writing a differential equation for $\sigma(-u)$, or otherwise, show that σ is an odd function.

(b) Show that $\sigma(u + \omega_i) = -\sigma(u)\exp(a_i(u + b_i))$ for some constants a_i, b_i. Use (a) to express b_i in terms of ω_i. [*Do not attempt to express a_i in terms of ω_i.*]

(c) Show that the function $f(u) = \sigma(2u)/\sigma(u)^4$ is periodic with respect to the lattice L and deduce that $f(u) = -\wp'(u)$.

10H Algebraic Curves

(a) Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety. Define the tangent space T_pX for $p \in X$. Show that the set

$$\{p \in X \mid \dim T_pX \geqslant d\}$$

is closed, for every $d \geqslant 0$.

(b) Let C be an irreducible projective curve, $p \in C$, and $f : C \backslash \{p\} \to \mathbb{P}^n$ a rational map. Show, carefully quoting any theorems that you use, that if C is smooth at p then f extends to a regular map at p.

11H Logic, Computation and Set Theory

(i) What does it mean for a function from $\mathbb{N}^k$ to $\mathbb{N}$ to be *recursive*? Write down a function that is not recursive. You should include a proof that your example is not recursive.

(ii) What does it mean for a subset of $\mathbb{N}^k$ to be *recursive*, and what does it mean for it to be *recursively enumerable*? Give, with proof, an example of a set that is recursively enumerable but not recursive. Prove that a set is recursive if and only if both it and its complement are recursively enumerable. If a set is recursively enumerable, must its complement be recursively enumerable?

[*You may assume the existence of any universal recursive functions or universal register machine programs that you wish.*]

12G Probability and Measure

Explain what is meant by the *characteristic function* ϕ of a real-valued random variable and prove that $|\phi|^2$ is also a characteristic function of some random variable.

Let us say that a characteristic function ϕ is *infinitely divisible* when, for each $n \geqslant 1$, we can write $\phi = (\phi_n)^n$ for some characteristic function ϕ_n. Prove that, in this case, the limit

$$\psi(t) = \lim_{n\to\infty} |\phi_{2n}(t)|^2$$

exists for all real t and is continuous at $t = 0$.

Using Lévy's continuity theorem for characteristic functions, which you should state carefully, deduce that ψ is a characteristic function. Hence show that, if ϕ is infinitely divisible, then $\phi(t)$ cannot vanish for any real t.

13I Applied Probability

State the product theorem for Poisson random measures.

Consider a system of n queues, each with infinitely many servers, in which, for $i = 1, \ldots, n-1$, customers leaving the ith queue immediately arrive at the $(i+1)$th queue. Arrivals to the first queue form a Poisson process of rate λ. Service times at the ith queue are all independent with distribution F, and independent of service times at other queues, for all i. Assume that initially the system is empty and write $V_i(t)$ for the number of customers at queue i at time $t \geqslant 0$. Show that $V_1(t), \ldots, V_n(t)$ are independent Poisson random variables.

In the case $F(t) = 1 - e^{-\mu t}$ show that

$$\mathbb{E}(V_i(t)) = \frac{\lambda}{\mu}\mathbb{P}(N_t \geqslant i), \quad t \geqslant 0, \quad i = 1, \ldots, n\ ,$$

where $(N_t)_{t\geqslant 0}$ is a Poisson process of rate μ.

Suppose now that arrivals to the first queue stop at time T. Determine the mean number of customers at the ith queue at each time $t \geqslant T$.

14J Optimization and Control

State Pontryagin's Maximum Principle (PMP).

In a given lake the tonnage of fish, x, obeys

$$dx/dt = 0.001(50 - x)x - u\,, \quad 0 < x \leqslant 50\,,$$

where u is the rate at which fish are extracted. It is desired to maximize

$$\int_0^\infty u(t)e^{-0.03t}\,dt\,,$$

choosing $u(t)$ under the constraints $0 \leqslant u(t) \leqslant 1.4$, and $u(t) = 0$ if $x(t) = 0$. Assume the PMP with an appropriate Hamiltonian $H(x,u,t,\lambda)$. Now define $G(x,u,t,\eta) = e^{0.03t}H(x,u,t,\lambda)$ and $\eta(t) = e^{0.03t}\lambda(t)$. Show that there exists $\eta(t)$, $0 \leqslant t$ such that on the optimal trajectory u maximizes

$$G(x,u,t,\eta) = \eta[0.001(50 - x)x - u] + u,$$

and

$$d\eta/dt = 0.002(x - 10)\eta\,.$$

Suppose that $x(0) = 20$ and that under an optimal policy it is not optimal to extract all the fish. Argue that $\eta(0) \geqslant 1$ is impossible and describe qualitatively what must happen under the optimal policy.

15I Principles of Statistics

(i) Let $X_1, \ldots, X_n$ be independent, identically distributed random variables, with the exponential density $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$.

Obtain the maximum likelihood estimator $\hat{\theta}$ of θ. What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

What is the minimum variance unbiased estimator of θ? Justify your answer carefully.

(ii) Explain briefly what is meant by the *profile log-likelihood* for a scalar parameter of interest γ, in the presence of a nuisance parameter ξ. Describe how you would test a null hypothesis of the form $H_0 : \gamma = \gamma_0$ using the profile log-likelihood ratio statistic.

In a reliability study, lifetimes $T_1, \ldots, T_n$ are independent and exponentially distributed, with means of the form $E(T_i) = \exp(\beta + \xi z_i)$ where β, ξ are unknown and $z_1, \ldots, z_n$ are known constants. Inference is required for the mean lifetime, $\exp(\beta + \xi z_0)$, for covariate value z_0.

Find, as explicitly as possible, the profile log-likelihood for $\gamma \equiv \beta + \xi z_0$, with nuisance parameter ξ.

Show that, under $H_0 : \gamma = \gamma_0$, the profile log-likelihood ratio statistic has a distribution which does not depend on the value of ξ. How might the parametric bootstrap be used to obtain a test of H_0 of exact size α?

[*Hint: if* Y *is exponentially distributed with mean* 1, *then* μY *is exponentially distributed with mean* μ.]

16J Stochastic Financial Models

(i) What does it mean to say that the process $(W_t)_{t \geqslant 0}$ is a *Brownian motion?* What does it mean to say that the process $(M_t)_{t \geqslant 0}$ is a *martingale?*

Suppose that $(W_t)_{t \geqslant 0}$ is a Brownian motion and the process $(X_t)_{t \geqslant 0}$ is given in terms of W as

$$X_t = x_0 + \sigma W_t + \mu t$$

for constants σ, μ. For what values of θ is the process

$$M_t = \exp(\theta X_t - \lambda t)$$

a martingale? (Here, λ is a positive constant.)

(ii) In a standard Black–Scholes model, the price at time t of a share is represented as $S_t = \exp(X_t)$. You hold a perpetual American put option on this share, with strike K; you may exercise at any stopping time τ, and upon exercise you receive $\max\{0, K - S_\tau\}$. Let $0 < a < \log K$. Suppose you plan to use the exercise policy: 'Exercise as soon as the price falls to e^a or lower.' Calculate what the option would be worth if you were to follow this policy. (Assume that the riskless rate of interest is constant and equal to $r > 0$.) For what choice of a is this value maximised?

17B Dynamical Systems

Let $f : I \to I$ be a continuous one-dimensional map of the interval $I \subset \mathbb{R}$. Explain what is meant by saying (a) that the map f is topologically transitive, and (b) that the map f has a horseshoe.

Consider the tent map defined on the interval $[0,1]$ by

$$f(x) = \begin{cases} \mu x & 0 \leqslant x < \frac{1}{2} \\ \mu(1-x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

for $1 < \mu \leqslant 2$. Show that if $\mu > \sqrt{2}$ then this map is topologically transitive, and also that f^2 has a horseshoe.

18D Partial Differential Equations

Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \tag{1}$$

to be solved for $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, subject to the initial conditions

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = 0 \tag{2}$$

for f in the Schwarz space $\mathcal{S}(\mathbb{R}^n)$. Use the Fourier transform in x to obtain a representation for the solution in the form

$$u(t,x) = \int e^{ix\cdot\xi} A(t,\xi) \widehat{f}(\xi) d^n\xi \tag{3}$$

where A should be determined explicitly. Explain carefully why your formula gives a smooth solution to (1) and why it satisfies the initial conditions (2), referring to the required properties of the Fourier transform as necessary.

Next consider the case $n = 1$. Find a tempered distribution T (depending on t, x) such that (3) can be written

$$u =< T, \widehat{f} >$$

and (using the definition of Fourier transform of tempered distributions) show that the formula reduces to

$$u(t,x) = \frac{1}{2}\big[f(x-t) + f(x+t)\big].$$

State and prove the Duhamel principle relating to the solution of the n-dimensional inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = h$$

to be solved for $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, subject to the initial conditions

$$u(0,x) = 0, \quad \frac{\partial u}{\partial t}(0,x) = 0$$

for h a C^∞ function. State clearly assumptions used on the solvability of the homogeneous problem.

[*Hint: it may be useful to consider the Fourier transform of the tempered distribution defined by the function* $\xi \mapsto e^{i\xi\cdot a}$.]

19D Methods of Mathematical Physics

Let

$$f(\lambda)=\int_\gamma e^{\lambda(t-t^3/3)}dt, \quad \lambda \text{ real and positive},$$

where γ is a path beginning at $\infty e^{-2i\pi/3}$ and ending at $+\infty$ (on the real axis). Identify the saddle points and sketch the paths of constant phase through these points.

Hence show that $f(\lambda)\sim e^{2\lambda/3}\sqrt{\pi/\lambda}$ as $\lambda\to\infty$.

20E Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right), \quad 0\leqslant x\leqslant 1, \quad t\geqslant 0,$$

with the initial condition $u(x,0)=\phi(x), 0\leqslant x\leqslant 1$ and zero boundary conditions at $x=0$ and $x=1$, is solved by the finite-difference method

$$u_m^{n+1}=u_m^n+\mu[a_{m-\frac{1}{2}}u_{m-1}^n-(a_{m-\frac{1}{2}}+a_{m+\frac{1}{2}})u_m^n+a_{m+\frac{1}{2}}u_{m+1}^n],$$
$$m=1,2,\ldots,N,$$

where $\mu=\Delta t/(\Delta x)^2$, $\quad \Delta x=\frac{1}{N+1}$ and $u_m^n\approx u(m\Delta x, n\Delta t)$, $a_\alpha=a(\alpha\Delta x)$.

Assuming sufficient smoothness of the function a, and that μ remains constant as $\Delta x>0$ and $\Delta t>0$ become small, prove that the exact solution satisfies the numerical scheme with error $O((\Delta x)^3)$.

(ii) For the problem defined in Part (i), assume that there exist $0<a_-<a_+<\infty$ such that $a_-\leqslant a(x)\leqslant a_+, \quad 0\leqslant x\leqslant 1$. Prove that the method is stable for $0<\mu\leqslant 1/(2a_+)$.

[*Hint: You may use without proof the Gerschgorin theorem: All the eigenvalues of the matrix* $A=(a_{k,l})_{k,l=1,\ldots,M}$ *are contained in* $\bigcup_{k=1}^m \mathbb{S}_k$, *where*

$$\mathbb{S}_k=\left\{z\in\mathbb{C}:|z-a_{k,k}|\leqslant\sum_{\substack{l=1\\l\neq k}}^m |a_{k,l}|\right\}, \quad k=1,2,\ldots,m. \quad]$$

21C Foundations of Quantum Mechanics

(i) What are the commutation relations satisfied by the components of an angular momentum vector $\mathbf{J}$? State the possible eigenvalues of the component J_3 when $\mathbf{J}^2$ has eigenvalue $j(j+1)\hbar^2$.

Describe how the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are used to construct the components of the angular momentum vector $\mathbf{S}$ for a spin $\frac{1}{2}$ system. Show that they obey the required commutation relations.

Show that S_1, S_2 and S_3 each have eigenvalues $\pm\frac{1}{2}\hbar$. Verify that $\mathbf{S}^2$ has eigenvalue $\frac{3}{4}\hbar^2$.

(ii) Let $\mathbf{J}$ and $|jm\rangle$ denote the standard operators and state vectors of angular momentum theory. Assume units where $\hbar = 1$. Consider the operator

$$U(\theta) = e^{-i\theta J_2}\,.$$

Show that

$$\begin{aligned} U(\theta)J_1U(\theta)^{-1} &= \cos\theta J_1 - \sin\theta J_3 \\ U(\theta)J_3U(\theta)^{-1} &= \sin\theta J_1 + \cos\theta J_3\,. \end{aligned}$$

Show that the state vectors $U\left(\frac{\pi}{2}\right)|jm\rangle$ are eigenvectors of J_1. Suppose that J_1 is measured for a system in the state $|jm\rangle$; show that the probability that the result is m' equals

$$|\langle jm'|e^{i\frac{\pi}{2}J_2}|jm\rangle|^2\,.$$

Consider the case $j = m = \frac{1}{2}$. Evaluate the probability that the measurement of J_1 will result in $m' = -\frac{1}{2}$.

22A Statistical Physics

A diatomic molecule, free to move in two space dimensions, has classical Hamiltonian

$$H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{1}{2I}J^2$$

where $\mathbf{p} = (p_1, p_2)$ is the particle's momentum and J is its angular momentum. Write down the classical partition function Z for an ideal gas of N such molecules in thermal equilibrium at temperature T. Show that it can be written in the form

$$Z = (z_t z_{rot})^N$$

where z_t and z_{rot} are the one-molecule partition functions associated with the translational and rotational degrees of freedom, respectively. Compute z_t and z_{rot} and hence show that the energy E of the gas is given by

$$E = \frac{3}{2}NkT$$

where k is Boltzmann's constant. How does this result illustrate the principle of equipartition of energy?

In an improved model of the two-dimensional gas of diatomic molecules, the angular momentum J is quantized in integer multiples of $\hbar$:

$$J = j\hbar\,, \qquad j = 0, \pm 1, \pm 2, \ldots .$$

Write down an expression for z_{rot} in this case. Given that $kT \ll (\hbar^2/2I)$, obtain an expression for the energy E in the form

$$E \approx AT + Be^{-\hbar^2/2IkT}$$

where A and B are constants that should be computed. How is this result compatible with the principle of equipartition of energy? Find C_v, the specific heat at constant volume, for $kT \ll (\hbar^2/2I)$.

Why can the sum over j in z_{rot} be approximated by an integral when $kT \gg (\hbar^2/2I)$? Deduce that $E \approx \frac{3}{2}NkT$ in this limit.

23C Applications of Quantum Mechanics

Consider the two Hamiltonians

$$H_1 = \frac{\mathbf{p}^2}{2m} + V(|\mathbf{r}|)\,,$$
$$H_2 = \frac{\mathbf{p}^2}{2m} + \sum_{n_i \in \mathbb{Z}} V(|\mathbf{r} - n_1\mathbf{a}_1 - n_2\mathbf{a}_2 - n_3\mathbf{a}_3|)\,,$$

where $\mathbf{a}_i$ are three linearly independent vectors. For each of the Hamiltonians $H = H_1$ and $H = H_2$, what are the symmetries of H and what unitary operators U are there such that $UHU^{-1} = H$?

For H_2 derive Bloch's theorem. Suppose that H_1 has energy eigenfunction $\psi_0(\mathbf{r})$ with energy E_0 where $\psi_0(\mathbf{r}) \sim Ne^{-Kr}$ for large $r = |\mathbf{r}|$. Assume that $K|\mathbf{a}_i| \gg 1$ for each i. In a suitable approximation derive the energy eigenvalues for H_2 when $E \approx E_0$. Verify that the energy eigenfunctions and energy eigenvalues satisfy Bloch's theorem. What happens if $K|\mathbf{a}_i| \to \infty$?

24B Fluid Dynamics II

A steady two-dimensional jet is generated in an infinite, incompressible fluid of density ρ and kinematic viscosity ν by a point source of momentum with momentum flux in the x direction F per unit length located at the origin.

Using boundary layer theory, analyse the motion in the jet and show that the x-component of the velocity is given by

$$u = U(x) f'(\eta),$$

where

$$\eta = y/\delta(x), \quad \delta(x) = (\rho\nu^2 x^2/F)^{1/3} \text{ and } U(x) = (F^2/\rho^2\nu x)^{1/3}\,.$$

Show that f satisfies the differential equation

$$f''' + \frac{1}{3}(ff'' + f'^2) = 0.$$

Write down the appropriate boundary conditions for this equation. [*You need not solve the equation.*]

25E Waves in Fluid and Solid Media

Derive the wave equation governing the velocity potential for linearised sound in a perfect gas. How is the pressure disturbance related to the velocity potential? Write down the spherically symmetric solution to the wave equation with time dependence $e^{i\omega t}$, which is regular at the origin.

A high pressure gas is contained, at density ρ_0, within a thin metal spherical shell which makes small amplitude spherically symmetric vibrations. Ignore the low pressure gas outside. Let the metal shell have radius a, mass m per unit surface area, and elastic stiffness which tries to restore the radius to its equilibrium value a_0 with a force $-\kappa(a-a_0)$ per unit surface area. Show that the frequency of these vibrations is given by

$$\omega^2 \left(m + \frac{\rho_0 a_0}{\theta \cot\theta - 1} \right) = \kappa \qquad \text{where } \theta = \omega a_0 / c_0 .$$