UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Wednesday 2 June 2003 9 to 12

PAPER 2

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **1D, 14D** *should be in one bundle and* **13I, 16I** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1D Principles of Dynamics

(i) The trajectory $\mathbf{x}(t)$ of a non-relativistic particle of mass m and charge q moving in an electromagnetic field obeys the Lorentz equation

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \frac{\dot{\mathbf{x}}}{c} \wedge \mathbf{B}) .$$

Show that this equation follows from the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\left(\phi - \frac{\dot{\mathbf{x}} \cdot \mathbf{A}}{c}\right)$$

where $\phi(\mathbf{x}, t)$ is the electromagnetic scalar potential and $\mathbf{A}(\mathbf{x}, t)$ the vector potential, so that

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \boldsymbol{\nabla}\phi \text{ and } \mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} .$$

(ii) Let $\mathbf{E} = 0$. Consider a particle moving in a constant magnetic field which points in the z direction. Show that the particle moves in a helix about an axis pointing in the z direction. Evaluate the radius of the helix.

2G Functional Analysis

(i) Define the dual of a normed vector space $(E, ||\cdot||)$. Show that the dual is always a complete normed space.

Prove that the vector space ℓ_1, consisting of those real sequences $(x_n)_{n=1}^{\infty}$ for which the norm

$$||(x_n)||_1 = \sum_{n=1}^{\infty} |x_n|$$

is finite, has the vector space ℓ_∞ of all bounded sequences as its dual.

(ii) State the Stone–Weierstrass approximation theorem.

Let K be a compact subset of $\mathbb{R}^n$. Show that every $f \in C_{\mathbb{R}}(K)$ can be uniformly approximated by a sequence of polynomials in n variables.

Let f be a continuous function on $[0,1] \times [0,1]$. Deduce that

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx\right) dy = \int_0^1 \left(\int_0^1 f(x,y)\, dy\right) dx .$$

3F Groups, Rings and Fields

(i) In each of the following two cases, determine a highest common factor in $\mathbb{Z}[i]$:

(a) $3+4i,\ 4-3i$;

(b) $3+4i,\ 1+2i$.

(ii) State and prove the Eisenstein criterion for irreducibility of polynomials with integer coefficients. Show that, if p is prime, the polynomial

$$1+x+\cdots+x^{p-1}$$

is irreducible over $\mathbb{Z}$.

4D Dynamics of Differential Equations

(i) What is a *Liapunov function*?

Consider the second order ODE

$$\dot{x}=y\ ,\quad \dot{y}=-y-\sin^3 x\,.$$

By finding a suitable Liapunov function of the form $V(x,y)=f(x)+g(y)$, where f and g are to be determined, show that the origin is asymptotically stable. Using your form of V, find the greatest value of y_0 such that a trajectory through $(0,y_0)$ is guaranteed to tend to the origin as $t\to\infty$.

[*Any theorems you use need not be proved but should be clearly stated.*]

(ii) Explain the use of the stroboscopic method for investigating the dynamics of equations of the form $\ddot{x}+x=\epsilon f(x,\dot{x},t)$, when $|\epsilon|\ll 1$. In particular, for $x=R\cos(t+\theta)$, $\dot{x}=-R\sin(t+\theta)$ derive the equations, correct to order ϵ,

$$\dot{R}=-\epsilon\langle f\sin(t+\theta)\rangle\ ,\quad R\dot{\theta}=-\epsilon\langle f\cos(t+\theta)\rangle, \qquad (*)$$

where the brackets denote an average over the period of the unperturbed oscillator.

Find the form of the right hand sides of these equations explicitly when $f=\Gamma x^2\cos t-3qx$, where $\Gamma>0$, $q\neq 0$. Show that apart from the origin there is another fixed point of $(*)$, and determine its stability. Sketch the trajectories in (R,θ) space in the case $q>0$. What do you deduce about the dynamics of the full equation?

[*You may assume that* $\langle\cos^2 t\rangle=\frac{1}{2}$, $\langle\cos^4 t\rangle=\frac{3}{8}$, $\langle\cos^2 t\sin^2 t\rangle=\frac{1}{8}$.]

5F Combinatorics

Prove Ramsey's theorem in its usual infinite form, namely, that if $\mathbb{N}^{(r)}$ is finitely coloured then there is an infinite subset $M \subset \mathbb{N}$ such that $M^{(r)}$ is monochromatic.

Now let the graph $\mathbb{N}^{(2)}$ be coloured with an infinite number of colours in such a way that there is no infinite $M \subset \mathbb{N}$ with $M^{(2)}$ monochromatic. By considering a suitable 2-colouring of the set $\mathbb{N}^{(4)}$ of 4-sets, show that there is an infinite $M \subset \mathbb{N}$ with the property that any two edges of $M^{(2)}$ of the form ad, bc with $a < b < c < d$ have different colours.

By considering two further 2-colourings of $\mathbb{N}^{(4)}$, show that there is an infinite $M \subset \mathbb{N}$ such that any two non-incident edges of $M^{(2)}$ have different colours.

6F Representation Theory

Let V_n be the space of homogeneous polynomials of degree n in two variables z_1 and z_2. Define a left action of $G = SU_2$ on the space of polynomials by setting

$$(gP)z = P(zg)\,,$$

where $P \in \mathbb{C}\,[z_1, z_2]$, $\quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z = (z_1, z_2) \quad$ and $\quad zg = (az_1 + cz_2, bz_1 + dz_2)$.

Show that

(a) the representations V_n are irreducible,

(b) the representations V_n exhaust the irreducible representations of SU_2, and

(c) the irreducible representations of SO_3 are given by $W_n = V_{2n} (n \geqslant 0)$.

7H Differentiable Manifolds

Let M and N be smooth manifolds. If $\pi : M \times N \to M$ is the projection onto the first factor and π^* is the map in cohomology induced by the pull-back map on differential forms, show that $\pi^*(H^k(M))$ is a direct summand of $H^k(M \times N)$ for each $k \geqslant 0$.

Taking $H^k(M)$ to be zero for $k < 0$ and $k > \dim M$, show that for $n \geqslant 1$ and all k

$$H^k(M \times S^n) \cong H^k(M) \oplus H^{k-n}(M).$$

[*You might like to use induction in n.*]

8G Algebraic Topology

Define the fundamental group of a topological space and explain briefly why a continuous map gives rise to a homomorphism between fundamental groups.

Let X be a subspace of the Euclidean space $\mathbb{R}^3$ which contains all of the points $(x, y, 0)$ with $(x, y) \neq (0, 0)$, and which does not contain any of the points $(0, 0, z)$. Show that X has an infinite fundamental group.

9G Number Fields

By Dedekind's theorem, or otherwise, factorise $2, 3, 5$ and 7 into prime ideals in the field $K = \mathbb{Q}(\sqrt{-34})$. Show that the ideal equations

$$[\omega] = [5, \omega][7, \omega] \ , \quad [\omega + 3] = [2, \omega + 3][5, \omega + 3]^2$$

hold in K, where $\omega = 1 + \sqrt{-34}$. Hence, prove that the ideal class group of K is cyclic of order 4.

[*It may be assumed that the Minkowski constant for K is $2/\pi$.*]

10H Algebraic Curves

(a) For which polynomials $f(x)$ of degree $d > 0$ does the equation $y^2 = f(x)$ define a smooth affine curve?

(b) Now let C be the completion of the curve defined in (a) to a projective curve. For which polynomials $f(x)$ of degree $d > 0$ is C a smooth projective curve?

(c) Suppose that C, defined in (b), is a smooth projective curve. Consider a map $p : C \to \mathbb{P}^1$, given by $p(x, y) = x$. Find the degree and the ramification points of p.

11H Logic, Computation and Set Theory

State the Axiom of Replacement.

Show that for any set x there is a transitive set y that contains x, indicating where in your argument you have used the Axiom of Replacement. No form of recursion theorem may be assumed without proof.

Which of the following are true and which are false? Give proofs or counterexamples as appropriate. You may assume standard properties of ordinals.

(a) If x is a transitive set then x is an ordinal.

(b) If each member of a set x is an ordinal then x is an ordinal.

(c) If x is a transitive set and each member of x is an ordinal then x is an ordinal.

12G Probability and Measure

Let H be a Hilbert space and let V be a closed subspace of H. Let $x \in H$. Show that there is a unique decomposition $x = u + v$ such that $v \in V$ and $u \in V^{\perp}$.

Now suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Define $\mathbb{E}(X|\mathcal{G})$ using a decomposition of the above type. Show that $\mathbb{E}(\mathbb{E}(X|\mathcal{G}).1_A) = \mathbb{E}(X.1_A)$ for each set $A \in \mathcal{G}$.

Let $\mathcal{G}_1 \subseteq \mathcal{G}_2$ be two sub-σ-algebras of $\mathcal{F}$. Show that

(a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) = \mathbb{E}(X|\mathcal{G}_1)$;

(b) $\mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(X|\mathcal{G}_1)$.

No general theorems about projections on Hilbert spaces may be quoted without proof.

13I Applied Probability

Let S_k be the sum of k independent exponential random variables of rate $k\mu$. Compute the moment generating function of S_k.

Consider, for each fixed k and for $0 < \lambda < \mu$, an $M/G/1$ queue with arrival rate λ and with service times distributed as S_k. Assume that the queue is empty at time 0 and write T_k for the earliest time at which a customer departs leaving the queue empty. Show that, as $k \to \infty$, T_k converges in distribution to a random variable T whose moment generating function $M_T(\theta)$ satisfies

$$\log\left(1 - \frac{\theta}{\lambda}\right) + \log M_T(\theta) = \left(\frac{\theta - \lambda}{\mu}\right)(1 - M_T(\theta)).$$

Hence obtain the mean value of T.

For what service-time distribution would the empty-to-empty time correspond exactly to T?

14J Information Theory

Let $\mathcal{X}$ be a binary linear code of length n, rank k and distance d. Let $x = (x_1, \ldots, x_n) \in \mathcal{X}$ be a codeword with exactly d non-zero digits.

(a) Prove that $n \geqslant d + k - 1$ (the Singleton bound).

(b) Prove that truncating $\mathcal{X}$ on the non-zero digits of x produces a code $\mathcal{X}'$ of length $n - d$, rank $k - 1$ and distance d' for some $d' \geqslant \lceil \frac{d}{2} \rceil$. Here $\lceil a \rceil$ is the integer satisfying $a \leqslant \lceil a \rceil < a + 1$, $a \in \mathbb{R}$.

[*Hint: Assume the opposite. Then, given $y \in \mathcal{X}$ and its truncation $y' \in \mathcal{X}'$, consider the coordinates where x and y have* 1 *in common (i.e. $x_j = y_j = 1$) and where they differ (e.g. $x_j = 1$ and $y_j = 0$).*]

(c) Deduce that $n \geqslant d + \sum_{1 \leqslant \ell \leqslant k-1} \lceil \frac{d}{2^\ell} \rceil$ (an improved Singleton bound).

15J Optimization and Control

The owner of a put option may exercise it on any one of the days $1, \ldots, h$, or not at all. If he exercises it on day t, when the share price is x_t, his profit will be $p - x_t$. Suppose the share price obeys $x_{t+1} = x_t + \epsilon_t$, where $\epsilon_1, \epsilon_2, \ldots$ are i.i.d. random variables for which $E|\epsilon_t| < \infty$. Let $F_s(x)$ be the maximal expected profit the owner can obtain when there are s further days to go and the share price is x. Show that

(a) $F_s(x)$ is non-decreasing in s,

(b) $F_s(x) + x$ is non-decreasing in x, and

(c) $F_s(x)$ is continuous in x.

Deduce that there exists a non-decreasing sequence, $a_1, \ldots, a_h$, such that expected profit is maximized by exercising the option the first day that $x_t \leqslant a_t$.

Now suppose that the option never expires, so effectively $h = \infty$. Show by examples that there may or may not exist an optimal policy of the form 'exercise the option the first day that $x_t \leqslant a$.'

16I Principles of Statistics

(i) Outline briefly the Bayesian approach to hypothesis testing based on Bayes factors.

(ii) Let Y_1, Y_2 be independent random variables, both uniformly distributed on $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. Find a minimal sufficient statistic for θ. Let $Y_{(1)} = \min\{Y_1, Y_2\}$, $Y_{(2)} = \max\{Y_1, Y_2\}$. Show that $R = Y_{(2)} - Y_{(1)}$ is ancilliary and explain why the Conditionality Principle would lead to inference about θ being drawn from the conditional distribution of $\frac{1}{2}\{Y_{(1)} + Y_{(2)}\}$ given R. Find the form of this conditional distribution.

17D Partial Differential Equations

(a) If f is a radial function on $\mathbb{R}^n$ (i.e. $f(x) = \phi(r)$ with $r = |x|$ for $x \in \mathbb{R}^n$) , and $n > 2$, then show that f is harmonic on $\mathbb{R}^n - \{0\}$ if and only if

$$\phi(r) = a + br^{2-n}$$

for $a, b \in \mathbb{R}$.

(b) State the mean value theorem for harmonic functions and prove it for $n > 2$.

(c) Generalise the statement and the proof of the mean value theorem to the case of a subharmonic function, i.e. a C^2 function such that $\Delta u \leqslant 0$.

18D Methods of Mathematical Physics

Let $\widehat{y}(p)$ be the Laplace transform of $y(t)$, where $y(t)$ satisfies

$$y'(t) = y(\pi - t)$$

and

$$y(0) = 1\,; \quad y(\pi) = k\,; \quad y(t) = 0 \text{ for } t < 0 \text{ and for } t > \pi\,.$$

Show that

$$p\widehat{y}(p) + ke^{-\pi p} - 1 = e^{-\pi p}\widehat{y}(-p)$$

and hence deduce that

$$\widehat{y}(p) = \frac{(k+p) - (1+pk)e^{-\pi p}}{1+p^2}\,.$$

Use the inversion formula for Laplace transforms to find $y(t)$ for $t > \pi$ and deduce that a solution of the above boundary value problem exists only if $k = -1$. Hence find $y(t)$ for $0 \leqslant t \leqslant \pi$.

19E Numerical Analysis

(i) Explain briefly what is meant by the *convergence* of a numerical method for ordinary differential equations.

(ii) Suppose the sufficiently-smooth function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ obeys the Lipschitz condition: there exists $\lambda > 0$ such that

$$||\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})|| \leqslant \lambda||\mathbf{x} - \mathbf{y}||, \qquad \mathbf{x},\mathbf{y} \in \mathbb{R}^d, t \geqslant 0\,.$$

Prove from first principles, without using the Dahlquist equivalence theorem, that the trapezoidal rule

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n,\mathbf{y}_n) + \mathbf{f}(t_{n+1},\mathbf{y}_{n+1})]$$

for the solution of the ordinary differential equation

$$\mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \quad t \geqslant 0, \quad \mathbf{y}(0) = \mathbf{y}_0,$$

converges.

20C Electrodynamics

A plane electromagnetic wave of frequency ω and wavevector $\mathbf{k}$ has an electromagnetic potential given by

$$A^a = A\epsilon^a e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

where A is the amplitude of the wave and ϵ^a is the polarization vector. Explain carefully why there are two independent polarization states for such a wave, and why $|\mathbf{k}|^2 = \omega^2$.

A wave travels in the positive z-direction with polarization vector $\epsilon^a = (0, 1, i, 0)$. It is incident at $z = 0$ on a plane surface which conducts perfectly in the x-direction, but not at all in the y-direction. Find an expression for the electromagnetic potential of the radiation that is reflected from this surface.

21C Foundations of Quantum Mechanics

(i) Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture. Explain how the two pictures provide equivalent descriptions of observable results.

Derive the equation of motion for an operator in the Heisenberg picture.

(ii) For a particle moving in one dimension, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, and the state vector is $|\Psi\rangle$. Eigenstates of $\hat{x}$ and $\hat{p}$ satisfy

$$\langle x|p\rangle = \left(\frac{1}{2\pi\hbar}\right)^{1/2} e^{ipx/\hbar}, \quad \langle x|x'\rangle = \delta(x-x'), \quad \langle p|p'\rangle = \delta(p-p')\,.$$

Use standard methods in the Dirac formalism to show that

$$\langle x|\hat{p}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x')$$
$$\langle p|\hat{x}|p'\rangle = i\hbar\frac{\partial}{\partial p}\delta(p-p')\,.$$

Calculate $\langle x|\hat{H}|x'\rangle$ and express $\langle x|\hat{p}|\Psi\rangle$, $\langle x|\hat{H}|\Psi\rangle$ in terms of the position space wave function $\Psi(x)$.

Compute the momentum space Hamiltonian for the harmonic oscillator with potential $V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$.

22C Applications of Quantum Mechanics

The Hamiltonian H_0 for a single electron atom has energy eigenstates $|\psi_n\rangle$ with energy eigenvalues E_n. There is an interaction with an electromagnetic wave of the form

$$H_1 = -e\,\mathbf{r}\cdot\boldsymbol{\epsilon}\,\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)\,, \qquad \omega = |\mathbf{k}|c\,,$$

where $\boldsymbol{\epsilon}$ is the polarisation vector. At $t=0$ the atom is in the state $|\psi_0\rangle$. Find a formula for the probability amplitude, to first order in e, to find the atom in the state $|\psi_1\rangle$ at time t. If the atom has a size a and $|\mathbf{k}|a \ll 1$ what are the selection rules which are relevant? For t large, under what circumstances will the transition rate be approximately constant?

[*You may use the result*

$$\int_{-\infty}^{\infty} \frac{\sin^2 \lambda t}{\lambda^2}\, d\lambda = \pi |t|\,. \quad]$$

23A General Relativity

(i) What is a "stationary" metric? What distinguishes a stationary metric from a "static" metric?

A Killing vector field K^a of a metric g_{ab} satisfies

$$K_{a;b} + K_{b;a} = 0.$$

Show that this is equivalent to

$$g_{ab,c}K^c + g_{ac}K^c{}_{,b} + g_{cb}K^c{}_{,a} = 0.$$

Hence show that a constant vector field K^a with one non-zero component, K^4 say, is a Killing vector field if g_{ab} is independent of x^4.

(ii) Given that K^a is a Killing vector field, show that $K_a u^a$ is constant along the geodesic worldline of a massive particle with 4-velocity u^a. Hence find the energy ε of a particle of unit mass moving in a static spacetime with metric

$$ds^2 = h_{ij}dx^i dx^j - e^{2U}dt^2,$$

where h_{ij} and U are functions only of the space coordinates x^i. By considering a particle with speed small compared with that of light, and given that $U \ll 1$, show that $h_{ij} = \delta_{ij}$ to lowest order in the Newtonian approximation, and that U is the Newtonian potential.

A metric admits an antisymmetric tensor Y_{ab} satisfying

$$Y_{ab;c} + Y_{ac;b} = 0.$$

Given a geodesic $x^a(\lambda)$, let $s_a = Y_{ab}\,\dot{x}^b$. Show that s_a is parallelly propagated along the geodesic, and that it is orthogonal to the tangent vector of the geodesic. Hence show that the scalar

$$\phi = s^a s_a$$

is constant along the geodesic.

24B Fluid Dynamics II

A plate is drawn vertically out of a bath and the resultant liquid drains off the plate as a thin film. Using lubrication theory, show that the governing equation for the thickness of the film, $h(x,t)$ is

$$\frac{\partial h}{\partial t} + \left(\frac{gh^2}{\nu}\right)\frac{\partial h}{\partial x} = 0, \qquad (*)$$

where t is time and x is the distance down the plate measured from the top.

Verify that

$$h(x,t) = F(x - \frac{gh^2}{\nu}t)$$

satisfies $(*)$ and identify the function $F(x)$. Using this relationship or otherwise, determine an explicit expression for the thickness of the film assuming that it was initially of uniform thickness h_0.

25E Waves in Fluid and Solid Media

Starting from the equations for the one-dimensional unsteady flow of a perfect gas of uniform entropy, derive the Riemann invariants

$$u \pm \frac{2}{\gamma - 1}c = \text{constant}$$

on characteristics

$$C_{\pm} : \frac{dx}{dt} = u \pm c.$$

A piston moves smoothly down a long tube, with position $x = X(t)$. Gas occupies the tube ahead of the piston, $x > X(t)$. Initially the gas and the piston are at rest, and the speed of sound in the gas is c_0. For $t > 0$, show that the C_+ characteristics are straight lines, provided that a shock-wave has not formed. Hence find a parametric representation of the solution for the velocity $u(x,t)$ of the gas.