UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Monday 2 June 2003 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **3F, 6F** *should be in one bundle and* **1J, 14J** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

(i) Let $(X_n, Y_n)_{n \geqslant 0}$ be a simple symmetric random walk in $\mathbb{Z}^2$, starting from $(0,0)$, and set $T = \inf\{n \geqslant 0 : \max\{|X_n|, |Y_n|\} = 2\}$. Determine the quantities $\mathbb{E}(T)$ and $\mathbb{P}(X_T = 2 \text{ and } Y_T = 0)$.

(ii) Let $(X_n)_{n \geqslant 0}$ be a discrete-time Markov chain with state-space I and transition matrix P. What does it mean to say that a state $i \in I$ is recurrent? Prove that i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, where $p_{ii}^{(n)}$ denotes the (i,i) entry in P^n.

Show that the simple symmetric random walk in $\mathbb{Z}^2$ is recurrent.

2D Principles of Dynamics

(i) Consider N particles moving in 3 dimensions. The Cartesian coordinates of these particles are $x^A(t),\ A = 1, \ldots, 3N$. Now consider an invertible change of coordinates to coordinates $q^a(x^A, t), \quad a = 1, \ldots, 3N$, so that one may express x^A as $x^A(q^a, t)$. Show that the velocity of the system in Cartesian coordinates $\dot{x}^A(t)$ is given by the following expression:

$$\dot{x}^A(\dot{q}^a, q^a, t) = \sum_{b=1}^{3N} \dot{q}^b \frac{\partial x^A}{\partial q^b}(q^a, t) + \frac{\partial x^A}{\partial t}(q^a, t)\,.$$

Furthermore, show that Lagrange's equations in the two coordinate systems are related via

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \sum_{A=1}^{3N} \frac{\partial x^A}{\partial q^a}\left(\frac{\partial L}{\partial x^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A}\right).$$

(ii) Now consider the case where there are $p < 3N$ constraints applied, $f^\ell(x^A, t) = 0,\ \ell = 1, \ldots, p$. By considering the $f^\ell,\ \ell = 1, \ldots, p$, and a set of independent coordinates $q^a, a = 1, \ldots, 3N - p$, as a set of $3N$ new coordinates, show that the Lagrange equations of the constrained system, i.e.

$$\frac{\partial L}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) + \sum_{\ell=1}^{p} \lambda^\ell \frac{\partial f^\ell}{\partial x^A} = 0, \quad A = 1, \ldots, 3N,$$
$$f^\ell = 0, \qquad \ell = 1, \ldots, p,$$

(where the λ^ℓ are Lagrange multipliers) imply Lagrange's equations for the unconstrained coordinates, i.e.

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0, \quad a = 1, \ldots, 3N - p\,.$$

3F Groups, Rings and Fields

(i) Let p be a prime number. Show that a group G of order p^n $(n \geqslant 2)$ has a nontrivial normal subgroup, that is, G is not a simple group.

(ii) Let p and q be primes, $p > q$. Show that a group G of order pq has a normal Sylow p-subgroup. If G has also a normal Sylow q-subgroup, show that G is cyclic. Give a necessary and sufficient condition on p and q for the existence of a non-abelian group of order pq. Justify your answer.

4A Electromagnetism

(i) Using Maxwell's equations as they apply to magnetostatics, show that the magnetic field $\mathbf{B}$ can be described in terms of a vector potential $\mathbf{A}$ on which the condition $\nabla \cdot \mathbf{A} = 0$ may be imposed. Hence derive an expression, valid at any point in space, for the vector potential due to a steady current distribution of density $\mathbf{j}$ that is non-zero only within a finite domain.

(ii) Verify that the vector potential $\mathbf{A}$ that you found in Part (i) satisfies $\nabla \cdot \mathbf{A} = 0$, and use it to obtain the Biot–Savart law expression for $\mathbf{B}$. What is the corresponding result for a steady surface current distribution of density $\mathbf{s}$?

In cylindrical polar coordinates (ρ, ϕ, z) (oriented so that $\mathbf{e}_\rho \times \mathbf{e}_\phi = \mathbf{e}_z$) a surface current

$$\mathbf{s} = s(\rho)\mathbf{e}_\phi$$

flows in the plane $z = 0$. Given that

$$s(\rho) = \begin{cases} 4I\left(1 + \frac{a^2}{\rho^2}\right)^{\frac{1}{2}} & a \leqslant \rho \leqslant 3a \\ 0 & \text{otherwise} \end{cases}$$

show that the magnetic field at the point $\mathbf{r} = a\mathbf{e}_z$ has z-component

$$B_z = \mu_0 I \log 5\,.$$

State, with justification, the full result for $\mathbf{B}$ at the point $\mathbf{r} = a\mathbf{e}_z$.

5F Combinatorics

Let G be a graph of order $n \geqslant 4$. Prove that if G has $t_2(n)+1$ edges then it contains two triangles with a common edge. Here, $t_2(n) = \lfloor n^2/4 \rfloor$ is the Turán number.

Suppose instead that G has exactly one triangle. Show that G has at most $t_2(n-1)+2$ edges, and that this number can be attained.

6F Representation Theory

Define the inner product $\langle \varphi, \psi \rangle$ of two class functions from the finite group G into the complex numbers. Prove that characters of the irreducible representations of G form an orthonormal basis for the space of class functions.

Consider the representation $\pi : S_n \to GL_n(\mathbb{C})$ of the symmetric group S_n by permutation matrices. Show that π splits as a direct sum $1 \oplus \rho$ where 1 denotes the trivial representation. Is the $(n-1)$-dimensional representation ρ irreducible?

7F Galois Theory

What does it mean to say that a field is *algebraically closed*? Show that a field M is algebraically closed if and only if, for any finite extension L/K and every homomorphism $\sigma : K \hookrightarrow M$, there exists a homomorphism $L \hookrightarrow M$ whose restriction to K is σ.

Let K be a field of characteristic zero, and M/K an algebraic extension such that every nonconstant polynomial over K has at least one root in M. Prove that M is algebraically closed.

8H Differentiable Manifolds

State the Implicit Function Theorem and outline how it produces submanifolds of Euclidean spaces.

Show that the unitary group $U(n) \subset GL(n, \mathbb{C})$ is a smooth manifold and find its dimension.

Identify the tangent space to $U(n)$ at the identity matrix as a subspace of the space of $n \times n$ complex matrices.

9G Number Fields

Let $K = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt[3]{10}$, and let $\mathcal{O}_K$ be the ring of algebraic integers of K. Show that the field polynomial of $r + s\alpha$, with r and s rational, is $(x-r)^3 - 10s^3$.

Let $\beta = \frac{1}{3}(\alpha^2 + \alpha + 1)$. By verifying that $\beta = 3/(\alpha - 1)$ and determining the field polynomial, or otherwise, show that β is in $\mathcal{O}_K$.

By computing the traces of $\theta, \alpha\theta, \alpha^2\theta$, show that the elements of $\mathcal{O}_K$ have the form

$$\theta = \tfrac{1}{3}(l + \tfrac{1}{10}m\alpha + \tfrac{1}{10}n\alpha^2)$$

where l, m, n are integers. By further computing the norm of $\frac{1}{10}\alpha(m + n\alpha)$, show that θ can be expressed as $\frac{1}{3}(u + v\alpha) + w\beta$ with u, v, w integers. Deduce that $1, \alpha, \beta$ form an integral basis for K.

10H Hilbert Spaces

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$.

(a) Define what it means for T to be (i) *invertible*, and (ii) *bounded below*. Prove that T is invertible if and only if both T and T^* are bounded below.

(b) Define what it means for T to be *normal*. Prove that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$. Deduce that, if T is normal, then every point of $\operatorname{Sp} T$ is an approximate eigenvalue of T.

(c) Let $S \in \mathcal{B}(H)$ be a self-adjoint operator, and let (x_n) be a sequence in H such that $\|x_n\| = 1$ for all n and $\|Sx_n\| \to \|S\|$ as $n \to \infty$. Show, by direct calculation, that

$$\left\|(S^2 - \|S\|^2)x_n\right\|^2 \to 0 \quad \text{as } n \to \infty,$$

and deduce that at least one of $\pm\|S\|$ is an approximate eigenvalue of S.

(d) Deduce that, with S as in (c),

$$r(S) = \|S\| = \sup\{|\langle Sx, x\rangle| : x \in H, \|x\| = 1\}\,.$$

11G Riemann Surfaces

Prove that a holomorphic map from $\mathbb{P}^1$ to itself is either constant or a rational function. Prove that a holomorphic map of degree 1 from $\mathbb{P}^1$ to itself is a Möbius transformation.

Show that, for every finite set of distinct points $z_1, z_2, \ldots, z_N$ in $\mathbb{P}^1$ and any values $w_1, w_2, \ldots, w_N \in \mathbb{P}^1$, there is a holomorphic function $f : \mathbb{P}^1 \to \mathbb{P}^1$ with $f(z_n) = w_n$ for $n = 1, 2, \ldots, N$.

12H Logic, Computation and Set Theory

(i) State Zorn's Lemma. Use Zorn's Lemma to prove that every real vector space has a basis.

(ii) State the Bourbaki–Witt Theorem, and use it to prove Zorn's Lemma, making clear where in the argument you appeal to the Axiom of Choice.

Conversely, deduce the Bourbaki–Witt Theorem from Zorn's Lemma.

If X is a non-empty poset in which every chain has an upper bound, must X be chain-complete?

13G Probability and Measure

State and prove the first Borel–Cantelli Lemma.

Suppose that (F_n) is a sequence of events in a common probability space such that $\mathbb{P}(F_i \cap F_j) \leqslant \mathbb{P}(F_i).\mathbb{P}(F_j)$ whenever $i \neq j$ and that $\sum_n \mathbb{P}(F_n) = \infty$.

Let 1_{F_n} be the indicator function of F_n and let

$$S_n = \sum_{k \leqslant n} 1_{F_k} \ ; \ \mu_n = \mathbb{E}(S_n)\,.$$

Use Chebyshev's inequality to show that

$$\mathbb{P}(S_n < \tfrac{1}{2}\mu_n) \leqslant \mathbb{P}(|S_n - \mu_n| > \tfrac{1}{2}\mu_n) \leqslant \tfrac{4}{\mu_n}\,.$$

Deduce, using the first Borel–Cantelli Lemma, that $\mathbb{P}(F_n \text{ infinitely often }) = 1$.

14J Information Theory

A binary Huffman code is used for encoding symbols $1, \ldots, m$ occurring with probabilities $p_1 \geqslant \ldots \geqslant p_m > 0$ where $\sum_{1 \leqslant j \leqslant m} p_j = 1$. Let s_1 be the length of a shortest codeword and s_m of a longest codeword. Determine the maximal and minimal values of s_1 and s_m, and find binary trees for which they are attained.

15I Principles of Statistics

(i) A public health official is seeking a rational policy of vaccination against a relatively mild ailment which causes absence from work. Surveys suggest that 60% of the population are already immune, but accurate tests to detect vulnerability in any individual are too costly for mass screening. A simple skin test has been developed, but is not completely reliable. A person who is immune to the ailment will have a negligible reaction to the skin test with probability 0.4, a moderate reaction with probability 0.5 and a strong reaction with probability 0.1. For a person who is vulnerable to the ailment the corresponding probabilities are 0.1, 0.4 and 0.5. It is estimated that the money-equivalent of work-hours lost from failing to vaccinate a vulnerable person is 20, that the unnecessary cost of vaccinating an immune person is 8, and that there is no cost associated with vaccinating a vulnerable person or failing to vaccinate an immune person. On the basis of the skin test, it must be decided whether to vaccinate or not. What is the Bayes decision rule that the health official should adopt?

(ii) A collection of I students each sit J exams. The ability of the ith student is represented by θ_i and the performance of the ith student on the jth exam is measured by X_{ij}. Assume that, given $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_I)$, an appropriate model is that the variables $\{X_{ij}\,,\ 1 \leqslant i \leqslant I,\ 1 \leqslant j \leqslant J\}$ are independent, and

$$X_{ij} \sim N(\theta_i, \tau^{-1}),$$

for a known positive constant τ. It is reasonable to assume, *a priori*, that the θ_i are independent with

$$\theta_i \sim N(\mu, \zeta^{-1}),$$

where μ and ζ are population parameters, known from experience with previous cohorts of students.

Compute the posterior distribution of $\boldsymbol{\theta}$ given the observed exam marks vector $\mathbf{X} = \{X_{ij}\,,\ 1 \leqslant i \leqslant I,\ 1 \leqslant j \leqslant J\}$.

Suppose now that τ is also unknown, but assumed to have a Gamma(α_0, β_0) distribution, for known α_0, β_0. Compute the posterior distribution of τ given $\boldsymbol{\theta}$ and $\mathbf{X}$. Find, up to a normalisation constant, the form of the marginal density of $\boldsymbol{\theta}$ given $\mathbf{X}$.

16J Stochastic Financial Models

(i) In the context of a single-period financial market with d traded assets, what is an *arbitrage?* What is an *equivalent martingale measure?*

A simple single-period financial market contains two assets, S^0 (a bond), and S^1 (a share). The period can be good, bad, or indifferent, with probabilities 1/3 each. At the beginning of the period, time 0, both assets are worth 1, i.e.

$$S_0^0 = 1 = S_0^1,$$

and at the end of the period, time 1, the share is worth

$$S_1^1 = \begin{cases} a & \text{if the period was bad,} \\ b & \text{if the period was indifferent,} \\ c & \text{if the period was good,} \end{cases}$$

where $a < b < c$. The bond is always worth 1 at the end of the period. Show that there is no arbitrage in this market if and only if $a < 1 < c$.

(ii) An agent with C^2 strictly increasing strictly concave utility U has wealth w_0 at time 0, and wishes to invest his wealth in shares and bonds so as to maximise his expected utility of wealth at time 1. Explain how the solution to his optimisation problem generates an equivalent martingale measure.

Assume now that $a = 3/4$, $b = 1$, and $c = 3/2$. Characterise all equivalent martingale measures for this problem. Characterise all equivalent martingale measures which arise as solutions of an agent's optimisation problem.

Calculate the largest and smallest possible prices for a European call option with strike 1 and expiry 1, as the pricing measure ranges over all equivalent martingale measures. Calculate the corresponding bounds when the pricing measure is restricted to the set arising from expected-utility-maximising agents' optimisation problems.

17B Dynamical Systems

Consider the one-dimensional map $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \mu x^2(1-x)$ with μ a real parameter. Find the range of values of μ for which the open interval $(0,1)$ is mapped into itself and contains at least one fixed point. Describe the bifurcation at $\mu = 4$ and find the parameter value for which there is a period-doubling bifurcation. Determine whether the fixed point is an attractor at this bifurcation point.

18D Partial Differential Equations

(a) Define characteristic hypersurfaces and state a local existence and uniqueness theorem for a quasilinear partial differential equation with data on a non-characteristic hypersurface.

(b) Consider the initial value problem

$$3u_x + u_y = -yu, \qquad u(x,0) = f(x),$$

for a function $u : \mathbb{R}^2 \to \mathbb{R}$ with C^1 initial data f given for $y = 0$. Obtain a formula for the solution by the method of characteristics and deduce that a C^1 solution exists for all $(x, y) \in \mathbb{R}^2$.

Derive the following (*well-posedness*) property for solutions $u(x, y)$ and $v(x, y)$ corresponding to data $u(x, 0) = f(x)$ and $v(x, 0) = g(x)$ respectively:

$$\sup_x |u(x,y) - v(x,y)| \leqslant \sup_x |f(x) - g(x)| \qquad \text{for all } y.$$

(c) Consider the initial value problem

$$3u_x + u_y = u^2, \qquad u(x,0) = f(x),$$

for a function $u : \mathbb{R}^2 \to \mathbb{R}$ with C^1 initial data f given for $y = 0$. Obtain a formula for the solution by the method of characteristics and hence show that if $f(x) < 0$ for all x, then the solution exists for all $y > 0$. Show also that if there exists x_0 with $f(x_0) > 0$, then the solution does not exist for all $y > 0$.

19D Methods of Mathematical Physics

By considering the integral

$$\int_C \left(\frac{t}{1-t}\right)^i dt\,,$$

where C is a large circle centred on the origin, show that

$$B(1+i, 1-i) = \pi \operatorname{cosech} \pi,$$

where

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt,\ \ Re(p) > 0, Re(q) > 0\,.$$

By using $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, deduce that $\Gamma(i)\,\Gamma(-i) = \pi \operatorname{cosech} \pi\,.$

20E Numerical Analysis

(i) The linear algebraic equations $A\mathbf{u} = \mathbf{b}$, where A is symmetric and positive-definite, are solved with the Gauss–Seidel method. Prove that the iteration always converges.

(ii) The Poisson equation $\nabla^2 u = f$ is given in the bounded, simply connected domain $\Omega \subseteq \mathbb{R}^2$, with zero Dirichlet boundary conditions on $\partial\Omega$. It is approximated by the five-point formula

$$U_{m-1,n} + U_{m,n-1} + U_{m+1,n} + U_{m,n+1} - 4U_{m,n} = (\Delta x)^2 f_{m,n},$$

where $U_{m,n} \approx u(m\Delta x, n\Delta x)$, $f_{m,n} = f(m\Delta x, n\Delta x)$, and $(m\Delta x, n\Delta x)$ is in the interior of Ω.

Assume for the sake of simplicity that the intersection of $\partial\Omega$ with the grid consists only of grid points, so that no special arrangements are required near the boundary. Prove that the method can be written in a vector notation, $A\mathbf{u} = \mathbf{b}$ with a negative-definite matrix A.

21C Electrodynamics

A particle of charge q and mass m moves non-relativistically with 4-velocity $u^a(t)$ along a trajectory $x^a(t)$. Its electromagnetic field is determined by the Liénard–Wiechert potential

$$A^a(\mathbf{x}', t') = \frac{q}{4\pi\epsilon_0} \frac{u^a(t)}{u_b(t)(x' - x(t))^b}$$

where $t' = t + |\mathbf{x} - \mathbf{x}'|$ and $\mathbf{x}$ denotes the spatial part of the 4-vector x^a.

Derive a formula for the Poynting vector at very large distances from the particle. Hence deduce Larmor's formula for the rate of loss of energy due to electromagnetic radiation by the particle.

A particle moves in the (x, y) plane in a constant magnetic field $\mathbf{B} = (0, 0, B)$. Initially it has kinetic energy E_0; derive a formula for the kinetic energy of this particle as a function of time.

22A Statistical Physics

A gas in equilibrium at temperature T and pressure P has quantum stationary states i with energies $E_i(V)$ in volume V. What does it mean to say that a change in volume from V to $V + dV$ is *reversible*?

Write down an expression for the probability that the gas is in state i. How is the entropy S defined in terms of these probabilities? Write down an expression for the energy E of the gas, and establish the relation

$$dE = TdS - PdV$$

for reversible changes.

By considering the quantity $F = E - TS$, derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V .$$

A gas obeys the equation of state

$$PV = RT + \frac{B(T)}{V}$$

where R is a constant and $B(T)$ is a function of T only. The gas is expanded isothermally, at temperature T, from volume V_0 to volume $2V_0$. Find the work ΔW done on the gas. Show that the heat ΔQ absorbed by the gas is given by

$$\Delta Q = RT \log 2 + \frac{T}{2V_0} \frac{dB}{dT}.$$

23C Applications of Quantum Mechanics

Define the differential cross section $\dfrac{d\sigma}{d\Omega}$. Show how it may be related to a scattering amplitude f defined in terms of the behaviour of a wave function ψ satisfying suitable boundary conditions as $r = |\mathbf{r}| \to \infty$.

For a particle scattering off a potential $V(r)$ show how $f(\theta)$, where θ is the scattering angle, may be expanded, for energy $E = \hbar^2 k^2/2m$, as

$$f(\theta) = \sum_{\ell=0}^{\infty} f_\ell(k) P_\ell(\cos\theta)\,,$$

and find $f_\ell(k)$ in terms of the phase shift $\delta_\ell(k)$. Obtain the optical theorem relating σ_{total} and $f(0)$.

Suppose that

$$e^{2i\delta_1} = \frac{E - E_0 - \frac{1}{2}i\Gamma}{E - E_0 + \frac{1}{2}i\Gamma}\,.$$

Why for $E \approx E_0$ may $f_1(k)$ be dominant, and what is the expected behaviour of $\dfrac{d\sigma}{d\Omega}$ for $E \approx E_0$?

[*For large* r

$$e^{ikr\cos\theta} \sim \frac{1}{2ikr} \sum_{\ell=0}^{\infty} (2\ell+1)\big((-1)^{\ell+1} e^{-ikr} + e^{ikr}\big) P_\ell(\cos\theta)\,.$$

Legendre polynomials satisfy

$$\int_{-1}^{1} P_\ell(t) P_{\ell'}(t)dt \;=\; \frac{2}{2\ell+1}\delta_{\ell\ell'}\,. \quad]$$

24A General Relativity

(i) The worldline $x^a(\lambda)$ of a massive particle moving in a spacetime with metric g_{ab} obeys the geodesic equation

$$\frac{d^2x^a}{d\tau^2} + \left\{ {}_b{}^a{}_c \right\} \frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

where τ is the particle's proper time and $\left\{ {}_b{}^a{}_c \right\}$ are the Christoffel symbols; these are the equations of motion for the Lagrangian

$$L_1 = -m\sqrt{-g_{ab}\dot{x}^a\dot{x}^b}$$

where m is the particle's mass, and $\dot{x}^a = dx^a/d\lambda$. Why is the choice of worldline parameter λ irrelevant? Among all possible worldlines passing through points A and B, why is $x^a(\lambda)$ the one that extremizes the proper time elapsed between A and B?

Explain how the equations of motion for a massive particle may be obtained from the alternative Lagrangian

$$L_2 = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b\,.$$

What can you conclude from the fact that L_2 has no explicit dependence on λ? How are the equations of motion for a massless particle obtained from L_2?

(ii) A photon moves in the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - \left(1 - \frac{2M}{r}\right) dt^2.$$

Given that the motion is confined to the plane $\theta = \pi/2$, obtain the radial equation

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right),$$

where E and h are constants, the physical meaning of which should be stated.

Setting $u = 1/r$, obtain the equation

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2.$$

Using the approximate solution

$$u = \frac{1}{b}\sin\phi + \frac{M}{2b^2}\left(3 + \cos 2\phi\right) + \ldots,$$

obtain the standard formula for the deflection of light passing far from a body of mass M with impact parameter b. Reinstate factors of G and c to give your result in physical units.

25B Fluid Dynamics II

Consider a two-dimensional horizontal vortex sheet of strength U at height h above a horizontal rigid boundary at $y = 0$, so that the inviscid fluid velocity is

$$\mathbf{u} = \begin{cases} (U, 0) & 0 < y < h \\ (0, 0) & y > h. \end{cases}$$

Examine the temporal linear instabililty of the sheet and determine the relevant dispersion relationship.

For what wavelengths is the sheet unstable?

Evaluate the temporal growth rate and the wave propagation speed in the limit of both short and long waves. Comment briefly on the significance of your results.

26E Waves in Fluid and Solid Media

Consider the equation

$$\frac{\partial^2 \phi}{\partial t^2} + \alpha^2 \frac{\partial^4 \phi}{\partial x^4} + \beta^2 \phi = 0, \qquad (*)$$

with α and β real constants. Find the dispersion relation for waves of frequency ω and wavenumber k. Find the phase velocity $c(k)$ and the group velocity $c_g(k)$, and sketch the graphs of these functions.

By multiplying $(*)$ by $\partial \phi / \partial t$, obtain an energy equation in the form

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where E represents the energy density and F the energy flux.

Now let $\phi(x, t) = A \cos(kx - \omega t)$, where A is a real constant. Evaluate the average values of E and F over a period of the wave to show that

$$\langle F \rangle = c_g \langle E \rangle.$$

Comment on the physical meaning of this result.