UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Friday 6 June 2003 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than **FOUR** *questions.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **2D, 6D** *should be in one bundle and* **11I, 14I** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

Consider a pack of cards labelled $1, \ldots, 52$. We repeatedly take the top card and insert it uniformly at random in one of the 52 possible places, that is, either on the top or on the bottom or in one of the 50 places inside the pack. How long on average will it take for the bottom card to reach the top?

Let p_n denote the probability that after n iterations the cards are found in increasing order. Show that, irrespective of the initial ordering, p_n converges as $n \to \infty$, and determine the limit p. You should give precise statements of any general results to which you appeal.

Show that, at least until the bottom card reaches the top, the ordering of cards inserted beneath it is uniformly random. Hence or otherwise show that, for all n,

$$|p_n - p| \leqslant 52(1 + \log 52)/n \ .$$

2D Principles of Dynamics

The action S of a Hamiltonian system may be regarded as a function of the final coordinates $q^a, a = 1, \ldots, N$, and the final time t by setting

$$S(q^a, t) = \int_{(q_i^a, t_i)}^{(q^a, t)} dt' [p^a(t')\dot{q}^a(t') - H(p^a(t'), q^a(t'), t')]$$

where the initial coordinates q_i^a and time t_i are held fixed, and $p^a(t'), q^a(t')$ are the solutions to Hamilton's equations with Hamiltonian H, satisfying $q^a(t) = q^a, q^a(t_i) = q_i^a$.

(a) Show that under an infinitesimal change of the final coordinates δq^a and time δt, the change in S is

$$\delta S = p_a(t)\delta q_a - H(p^a(t), q^a(t), t)\delta t\,.$$

(b) Hence derive the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t}(q^a, t) + H\left(\frac{\partial S}{\partial q^a}(q^a, t), q^a, t\right) = 0\,. \qquad (*)$$

(c) If we can find a solution to $(*)$,

$$S = S(q^a, t; P^a),$$

where P^a are N integration constants, then we can use S as a generating function of type II, where

$$p^a = \frac{\partial S}{\partial q^a} \quad , \quad Q^a = -\frac{\partial S}{\partial P^a}\,.$$

Show that the Hamiltonian K in the new coordinates Q^a, P^a vanishes.

(d) Write down and solve the Hamilton–Jacobi equation for the one-dimensional simple harmonic oscillator, where $H = \frac{1}{2}(p^2 + q^2)$. Show the solution takes the form $S(q, t; E) = W(q, E) - Et$. Using this as a generating function $F_{II}(q, t, P)$ show that the new coordinates Q, P are constants of the motion and give their physical interpretation.

3G Functional Analysis

(i) State the Monotone Convergence Theorem and explain briefly how to prove it.

(ii) For which real values of α is $x^{-\alpha}\log x \in L^1((1,\infty))$?

Let $p > 0$. Using the Monotone Convergence Theorem and the identity

$$\frac{1}{x^p(x-1)} = \sum_{n=0}^{\infty} \frac{1}{x^{p+n+1}}$$

prove carefully that

$$\int_1^{\infty} \frac{\log x}{x^p(x-1)}\, dx = \sum_{n=0}^{\infty} \frac{1}{(n+p)^2}\,.$$

4F Groups, Rings and Fields

Write an essay on the theory of invariants. Your essay should discuss the theorem on the finite generation of the ring of invariants, the theorem on elementary symmetric functions, and some examples of calculation of rings of invariants.

5A Electromagnetism

Let $\mathbf{E}(\mathbf{r})$ be the electric field due to a continuous static charge distribution $\rho(\mathbf{r})$ for which $|\mathbf{E}| \to 0$ as $|\mathbf{r}| \to \infty$. Starting from consideration of a finite system of point charges, deduce that the electrostatic energy of the charge distribution ρ is

$$W = \frac{1}{2}\varepsilon_0 \int |\mathbf{E}|^2 d\tau \qquad (*)$$

where the volume integral is taken over all space.

A sheet of perfectly conducting material in the form of a surface S, with unit normal $\mathbf{n}$, carries a surface charge density σ. Let $E_\pm = \mathbf{n}\cdot\mathbf{E}_\pm$ denote the normal components of the electric field $\mathbf{E}$ on either side of S. Show that

$$\frac{1}{\varepsilon_0}\sigma = E_+ - E_-\,.$$

Three concentric spherical shells of perfectly conducting material have radii a, b, c with $a < b < c$. The innermost and outermost shells are held at zero electric potential. The other shell is held at potential V. Find the potentials $\phi_1(r)$ in $a < r < b$ and $\phi_2(r)$ in $b < r < c$. Compute the surface charge density σ on the shell of radius b. Use the formula $(*)$ to compute the electrostatic energy of the system.

6D Dynamics of Differential Equations

Explain what is meant by a *steady-state bifurcation* of a fixed point $\mathbf{x}_0(\mu)$ of an ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$, in $\mathbb{R}^n$, where μ is a real parameter. Give examples for $n = 1$ of equations exhibiting saddle-node, transcritical and pitchfork bifurcations.

Consider the system in $\mathbb{R}^2$, with $\mu > 0$,

$$\dot{x} = x(1 - y - 4x^2) \ , \quad \dot{y} = y(\mu - y - x^2) \ .$$

Show that the fixed point $(0, \mu)$ has a bifurcation when $\mu = 1$, while the fixed points $(\pm\frac{1}{2}, 0)$ have a bifurcation when $\mu = \frac{1}{4}$. By finding the first approximation to the extended centre manifold, construct the normal form at the bifurcation point in each case, and determine the respective bifurcation types. Deduce that for μ just greater than $\frac{1}{4}$, and for μ just less than 1, there is a stable pair of "mixed-mode" solutions with $x^2 > 0$, $y > 0$.

7H Geometry of Surfaces

Write an essay on the *Theorema Egregium* for surfaces in $\mathbb{R}^3$.

8H Logic, Computation and Set Theory

Write an essay on propositional logic. You should include all relevant definitions, and should cover the Completeness Theorem, as well as the Compactness Theorem and the Decidability Theorem.

[*You may assume that the set of primitive propositions is countable. You do not need to give proofs of simple examples of syntactic implication, such as the fact that $p \Rightarrow p$ is a theorem or that $p \Rightarrow q$ and $q \Rightarrow r$ syntactically imply $p \Rightarrow r$.*]

9F Graph Theory

Write an essay on the vertex-colouring of graphs drawn on compact surfaces other than the sphere. You should include a proof of Heawood's bound, and an example of a surface for which this bound is not attained.

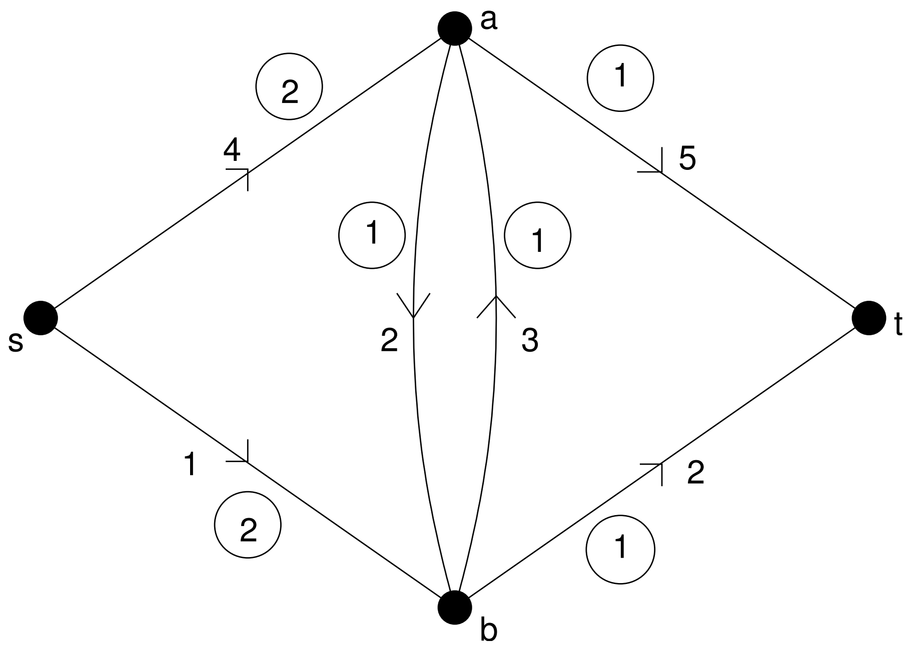
10G Number Theory

Write an essay describing the factor base method for factorising a large odd positive integer n. Your essay should include a detailed explanation of how the continued fraction of $\sqrt{n}$ can be used to find a suitable factor base.

11I Algorithms and Networks

Define the optimal distribution problem. State what it means for a circuit P to be flow-augmenting, and what it means for P to be unbalanced. State the optimality theorem for flows. Describe the simplex-on-a-graph algorithm, giving a brief justification of its stopping rules.

Consider the problem of finding, in the network shown below, a minimum-cost flow from s to t of value 2. Here the circled numbers are the upper arc capacities, the lower arc capacities all being zero, and the uncircled numbers are costs. Apply the simplex-on-a-graph algorithm to solve this problem, taking as initial flow the superposition of a unit flow along the path $s, (s, a), a, (a, t), t$ and a unit flow along the path $s, (s, a), a, (a, b), b, (b, t), t$.



12J Stochastic Financial Models

A single-period market contains d risky assets, $S^1, S^2, \ldots, S^d$, initially worth $(S_0^1, S_0^2, \ldots, S_0^d)$, and at time 1 worth random amounts $(S_1^1, S_1^2, \ldots, S_1^d)$ whose first two moments are given by

$$\mu = ES_1, \quad V = \text{cov}(S_1) \equiv E\big[\,(S_1 - ES_1)(S_1 - ES_1)^T\,\big].$$

An agent with given initial wealth w_0 is considering how to invest in the available assets, and has asked for your advice. Develop the theory of the mean-variance efficient frontier far enough to exhibit explicitly the minimum-variance portfolio achieving a required mean return, assuming that V is non-singular. How does your analysis change if a riskless asset S^0 is added to the market? Under what (sufficient) conditions would an agent maximising expected utility actually choose a portfolio on the mean-variance efficient frontier?

13I Principles of Statistics

Write an account, with appropriate examples, of inference in multiparameter exponential families. Your account should include a discussion of natural statistics and their properties and of various conditional tests on natural parameters.

14I Computational Statistics and Statistical Modelling

The nave height x, and the nave length y for 16 Gothic-style cathedrals and 9 Romanesque-style cathedrals, all in England, have been recorded, and the corresponding R output (slightly edited) is given below.

```
> first.lm _ lm(y ~ x + Style); summary(first.lm)

Call:

lm(formula = y ~ x + Style)


Residuals:

    Min      1Q  Median      3Q     Max

-172.67  -30.44   20.38   55.02   96.50


Coefficients:

            Estimate Std. Error t value Pr(>|t|)

(Intercept)   44.298     81.648   0.543   0.5929

x              4.712      1.058   4.452   0.0002

Style2        80.393     32.306   2.488   0.0209


Residual standard error: 77.53 on 22 degrees of freedom

Multiple R-Squared: 0.5384
```

You may assume that x, y are in suitable units, and that "style" has been set up as a factor with levels 1,2 corresponding to Gothic, Romanesque respectively.

(a) Explain carefully, with suitable graph(s) if necessary, the results of this analysis.

(b) Using the general model $Y = X\beta + \epsilon$ (in the conventional notation) explain carefully the theory needed for (a).

[*Standard theorems need not be proved.*]

15C Foundations of Quantum Mechanics

Discuss the quantum mechanics of the one-dimensional harmonic oscillator using creation and annihilation operators, showing how the energy levels are calculated.

A quantum mechanical system consists of two interacting harmonic oscillators and has the Hamiltonian

$$H = \frac{1}{2}\hat{p}_1^2 + \frac{1}{2}\hat{x}_1^2 + \frac{1}{2}\hat{p}_2^2 + \frac{1}{2}\hat{x}_2^2 + \lambda \hat{x}_1 \hat{x}_2 \,.$$

For $\lambda = 0$, what are the degeneracies of the three lowest energy levels? For $\lambda \neq 0$ compute, to lowest non-trivial order in perturbation theory, the energies of the ground state and first excited state.

[*Standard results for perturbation theory may be stated without proof.*]

16C Quantum Physics

Describe the energy band structure available to electrons moving in crystalline materials. How can it be used to explain the properties of crystalline materials that are conductors, insulators and semiconductors?

Where does the Fermi energy lie in an intrinsic semiconductor? Describe the process of doping of semiconductors and explain the difference between n-type and p-type doping. What is the effect of the doping on the position of the Fermi energy in the two cases?

Why is there a potential difference across a junction of n-type and p-type semiconductors?

Derive the relation

$$I = I_0 \left(1 - e^{-qV/kT}\right)$$

between the current, I, and the voltage, V, across an np junction, where I_0 is the total minority current in the semiconductor and q is the charge on the electron, T is the temperature and k is Boltzmann's constant. Your derivation should include an explanation of the terms *majority current* and *minority current*.

Why can the np junction act as a rectifier?

17A General Relativity

What are "inertial coordinates" and what is their physical significance? [*A proof of the existence of inertial coordinates is not required.*] Let O be the origin of inertial coordinates and let $R_{abcd}|_O$ be the curvature tensor at O (with all indices lowered). Show that $R_{abcd}|_O$ can be expressed entirely in terms of second partial derivatives of the metric g_{ab}, evaluated at O. Use this expression to deduce that

(a) $R_{abcd} = -R_{bacd}$

(b) $R_{abcd} = R_{cdab}$

(c) $R_{a[bcd]} = 0$.

Starting from the expression for $R^a{}_{bcd}$ in terms of the Christoffel symbols, show (again by using inertial coordinates) that

$R_{ab[cd;e]} = 0$.

Obtain the contracted Bianchi identities and explain why the Einstein equations take the form

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} - \Lambda g_{ab},$$

where T_{ab} is the energy-momentum tensor of the matter and Λ is an arbitrary constant.

18A Statistical Physics and Cosmology

Let $g(p)$ be the density of states of a particle in volume V as a function of the magnitude p of the particle's momentum. Explain why $g(p) \propto Vp^2/h^3$, where h is Planck's constant. Write down the Bose–Einstein and Fermi–Dirac distributions for the (average) number $\bar{n}(p)$ of particles of an ideal gas with momentum p. Hence write down integrals for the (average) total number N of particles and the (average) total energy E as functions of temperature T and chemical potential μ. Why do N and E also depend on the volume V?

Electromagnetic radiation in thermal equilibrium can be regarded as a gas of photons. Why are photons "ultra-relativistic" and how is photon momentum p related to the frequency ν of the radiation? Why does a photon gas have zero chemical potential? Use your formula for $\bar{n}(p)$ to express the energy density ε_γ of electromagnetic radiation in the form

$$\varepsilon_\gamma = \int_0^\infty \epsilon(\nu) d\nu$$

where $\epsilon(\nu)$ is a function of ν that you should determine up to a dimensionless multiplicative constant. Show that $\epsilon(\nu)$ is independent of h when $kT \gg h\nu$, where k is Boltzmann's constant. Let ν_{peak} be the value of ν at the maximum of the function $\epsilon(\nu)$; how does ν_{peak} depend on T?

Let n_γ be the photon number density at temperature T. Show that $n_\gamma \propto T^q$ for some power q, which you should determine. Why is n_γ unchanged as the volume V is increased quasi-statically? How does T depend on V under these circumstances? Applying your result to the Cosmic Microwave Background Radiation (CMBR), deduce how the temperature T_γ of the CMBR depends on the scale factor a of the Universe. At a time when $T_\gamma \sim 3000K$, the Universe underwent a transition from an earlier time at which it was opaque to a later time at which it was transparent. Explain briefly the reason for this transition and its relevance to the CMBR.

An ideal gas of fermions f of mass m is in equilibrium at temperature T and chemical potential μ_f with a gas of its own anti-particles $\bar{f}$ and photons (γ). Assuming that chemical equilibrium is maintained by the reaction

$$f + \bar{f} \leftrightarrow \gamma$$

determine the chemical potential $\mu_{\bar{f}}$ of the antiparticles. Let n_f and $n_{\bar{f}}$ be the number densities of f and $\bar{f}$, respectively. What will their values be for $kT \ll mc^2$ if $\mu_f = 0$? Given that $\mu_f > 0$, but $\mu_f \ll kT$, show that

$$n_f \approx n_0(T)\left[1 + \frac{\mu_f}{kT} F\left(mc^2/kT\right)\right]$$

where $n_0(T)$ is the fermion number density at zero chemical potential and F is a positive function of the dimensionless ratio mc^2/kT. What is F when $kT \ll mc^2$?

Given that $\mu_f \ll kT$, obtain an expression for the ratio $(n_f - n_{\bar{f}})/n_0$ in terms of μ, T and the function F. Supposing that f is either a proton or neutron, why should you expect the ratio $(n_f - n_{\bar{f}})/n_\gamma$ to remain constant as the Universe expands?

19E Transport Processes

A shallow layer of fluid of viscosity μ, density ρ and depth $h(x,t)$ lies on a rigid horizontal plane $y=0$ and is bounded by impermeable barriers at $x=-L$ and $x=L$ ($L \gg h$). Gravity acts vertically and a wind above the layer causes a shear stress $\tau(x)$ to be exerted on the upper surface in the $+x$ direction. Surface tension is negligible compared to gravity.

(a) Assuming that the steady flow in the layer can be analysed using lubrication theory, show that the horizontal pressure gradient p_x is given by $p_x = \rho g h_x$ and hence that

$$hh_x = \frac{3}{2}\frac{\tau}{\rho g}. \tag{1}$$

Show also that the fluid velocity at the surface $y=h$ is equal to $\tau h/4\mu$, and sketch the velocity profile for $0 \leqslant y \leqslant h$.

(b) In the case in which τ is a constant, τ_0, and assuming that the difference between h and its average value h_0 remains small compared with h_0, show that

$$h \approx h_0\left(1 + \frac{3\tau_0 x}{2\rho g h_0^2}\right)$$

provided that

$$\frac{\tau_0 L}{\rho g h_0^2} \ll 1.$$

(c) Surfactant at surface concentration $\Gamma(x)$ is added to the surface, so that now

$$\tau = \tau_0 - A\Gamma_x, \tag{2}$$

where A is a positive constant. The surfactant is advected by the surface fluid velocity and also experiences a surface diffusion with diffusivity D. Write down the equation for conservation of surfactant, and hence show that

$$(\tau_0 - A\Gamma_x)\, h\Gamma = 4\mu D\Gamma_x. \tag{3}$$

From equations (1), (2) and (3) deduce that

$$\frac{\Gamma}{\Gamma_0} = \exp\left[\frac{\rho g}{18\mu D}\left(h^3 - h_0^3\right)\right],$$

where Γ_0 is a constant. Assuming once more that $h_1 \equiv h - h_0 \ll h_0$, and that $h = h_0$ at $x = 0$, show further that

$$h_1 \approx \frac{3\tau_0 x}{2\rho g h_0}\left[1 + \frac{A\Gamma_0 h_0}{4\mu D}\right]^{-1}$$

provided that

$$\frac{\tau_0 h_0 L}{\mu D} \ll 1 \quad \text{as well as} \quad \frac{\tau_0 L}{\rho g h_0^2} \ll 1.$$

20E Theoretical Geophysics

Define the Rossby number. Under what conditions will a fluid flow be at (a) high and (b) low values of the Rossby number? Briefly describe both an oceanographic and a meteorological example of each type of flow.

Explain the concept of quasi-geostrophy for a thin layer of homogeneous fluid in a rapidly rotating system. Write down the quasi-geostrophic approximation for the vorticity in terms of the pressure, the fluid density and the rate of rotation. Define the potential vorticity and state the associated conservation law.

A broad current flows directly eastwards ($+x$ direction) with uniform velocity U across a flat ocean basin of depth H. The current encounters a low, two-dimensional ridge of width L and height $Hh(x)$ $(0 < x < L)$, whose axis is aligned in the north-south (y) direction. Neglecting any effects of stratification and assuming a constant vertical rate of rotation $\frac{1}{2}f$, such that the Rossby number is small, determine the effect of the ridge on the current. Show that the direction of the current after it leaves the ridge is dependent on the cross-sectional area of the ridge, but not on the explicit form of $h(x)$.

21B Mathematical Methods

Let $y(x,\lambda)$ denote the solution for $0 \leqslant x < \infty$ of

$$\frac{d^2y}{dx^2} - (x+\lambda^2)y = 0,$$

subject to the conditions that $y(0,\lambda) = a$ and $y(x,\lambda)\to 0$ as $x\to\infty$, where $a > 0$; it may be assumed that $y(x,\lambda) > 0$ for $x > 0$. Write $y(x,\lambda)$ in the form

$$y(x,\lambda) = \exp(z(x,\lambda))\,,$$

and consider an asymptotic expansion of the form

$$z(x,\lambda) \sim \sum_{n=0}^{\infty} \lambda^{1-n}\phi_n(x)\,,$$

valid in the limit $\lambda\to\infty$ with $x = O(1)$. Find $\phi_0(x), \phi_1(x), \phi_2(x)$ and $\phi_3(x)$.

It is known that the solution $y(x,\lambda)$ is of the form

$$y(x,\lambda) = c\,Y(X)$$

where

$$X = x + \lambda^2$$

and the constant factor c depends on λ. By letting $Y(X) = \exp(Z(X))$, show that the expression

$$Z(X) = -\frac{2}{3}X^{3/2} - \frac{1}{4}\ln X$$

satisfies the relevant differential equation with an error of $O(1/X^{3/2})$ as $X \to \infty$. Comment on the relationship between your answers for $z(x,\lambda)$ and $Z(X)$.

22B Nonlinear Waves and Integrable Systems

Let $\Phi^+(t), \Phi^-(t)$ denote the boundary values of functions which are analytic inside and outside a disc of radius $\frac{1}{2}$ centred at the origin. Let C denote the boundary of this disc.

Suppose that Φ^+, Φ^- satisfy the jump condition

$$\Phi^+(t) = \frac{t}{t^2-1}\Phi^-(t) + \frac{t^3-t^2+1}{t^2-t}, \quad t \in C\,.$$

(a) Show that the associated index is 1.

(b) Find the canonical solution of the homogeneous problem, i.e. the solution satisfying

$$X(z) \sim z^{-1}, \quad z \to \infty\,.$$

(c) Find the general solution of the Riemann–Hilbert problem satisfying the above jump condition as well as

$$\Phi(z) = O(z^{-1}), \quad z \to \infty\,.$$

(d) Use the above result to solve the linear singular integral problem

$$(t^2+t-1)\phi(t) + \frac{t^2-t-1}{\pi i}\oint_C \frac{\phi(\tau)}{\tau - t}\,d\tau = \frac{2(t^3-t^2+1)(t+1)}{t}, \quad t \in C\,.$$

23E Numerical Analysis

Write an essay on the conjugate gradient method. Your essay should include:

(a) a statement of the method and a sketch of its derivation;

(b) discussion, without detailed proofs, but with precise statements of relevant theorems, of the conjugacy of the search directions;

(c) a description of the standard form of the algorithm;

(d) discussion of the connection of the method with Krylov subspaces.