UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 6 June 2003 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** $\ldots$**, H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Analysis II

Explain what it means for a sequence of functions (f_n) to converge uniformly to a function f on an interval. If (f_n) is a sequence of continuous functions converging uniformly to f on a finite interval $[a,b]$, show that

$$\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx \qquad \text{as } n \to \infty\ .$$

Let $f_n(x) = x\exp(-x/n)/n^2,\ x \geqslant 0$. Does $f_n \to 0$ uniformly on $[0,\infty)$? Does $\int_0^\infty f_n(x)\,dx \to 0$? Justify your answers.

2D Methods

Consider the wave equation in a spherically symmetric coordinate system

$$\frac{\partial^2 u(r,t)}{\partial t^2} = c^2 \Delta u(r,t)\,,$$

where $\Delta u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$ is the spherically symmetric Laplacian operator.

(a) Show that the general solution to the equation above is

$$u(r,t) = \tfrac{1}{r}\left[f(r+ct) + g(r-ct)\right],$$

where $f(x), g(x)$ are arbitrary functions.

(b) Using separation of variables, determine the wave field $u(r,t)$ in response to a pulsating source at the origin $u(0,t) = A\,\sin\omega t$.

3H Statistics

The following table contains a distribution obtained in 320 tosses of 6 coins and the corresponding expected frequencies calculated with the formula for the binomial distribution for $p = 0.5$ and $n = 6$.

No. heads	0	1	2	3	4	5	6
Observed frequencies	3	21	85	110	62	32	7
Expected frequencies	5	30	75	100	75	30	5

Conduct a goodness-of-fit test at the 0.05 level for the null hypothesis that the coins are all fair.

[*Hint:*

Distribution	χ^2_5	χ^2_6	χ^2_7
95% percentile	11.07	12.59	14.07

]

4E Further Analysis

(a) State and prove Morera's Theorem.

(b) Let D be a domain and for each $n \in \mathbb{N}$ let $f_n : D \to \mathbb{C}$ be an analytic function. Suppose that $f : D \to \mathbb{C}$ is another function and that $f_n \to f$ uniformly on D. Prove that f is analytic.

5H Optimization

State and prove the Lagrangian sufficiency theorem for a general optimization problem with constraints.

6G Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U such that $\alpha^2 = \alpha$. Show that U can be written as the direct sum of the kernel of α and the image of α. Hence or otherwise, find the characteristic polynomial of α in terms of the dimension of U and the rank of α. Is α diagonalizable? Justify your answer.

7C Fluid Dynamics

Inviscid fluid issues vertically downwards at speed u_0 from a circular tube of radius a. The fluid falls onto a horizontal plate a distance H below the end of the tube, where it spreads out axisymmetrically.

Show that while the fluid is falling freely it has speed

$$u = u_0 \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{1/2},$$

and occupies a circular jet of radius

$$R = a \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{-1/4},$$

where z is the height above the plate and g is the acceleration due to gravity.

Show further that along the plate, at radial distances $r \gg a$ (i.e. far from the falling jet), where the fluid is flowing almost horizontally, it does so as a film of height $h(r)$, where

$$\frac{a^4}{4r^2h^2} = 1 + \frac{2g}{u_0^2}(H - h).$$

8B Complex Methods

Find the Laurent series centred on 0 for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in each of the domains

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$.

9A Special Relativity

Prove that the two-dimensional Lorentz transformation can be written in the form

$$\begin{aligned} x' &= x\cosh\phi - ct\sinh\phi \\ ct' &= -x\sinh\phi + ct\cosh\phi, \end{aligned}$$

where $\tanh\phi = v/c$. Hence, show that

$$\begin{aligned} x' + ct' &= e^{-\phi}(x + ct) \\ x' - ct' &= e^{\phi}(x - ct)\,. \end{aligned}$$

Given that frame S' has speed v with respect to S and S'' has speed v' with respect to S', use this formalism to find the speed v'' of S'' with respect to S.

[*Hint: rotation through a hyperbolic angle* ϕ*, followed by rotation through* ϕ'*, is equivalent to rotation through* $\phi + \phi'$.]

SECTION II

10F Analysis II

Let $(f_n)_{n \geqslant 1}$ be a sequence of continuous complex-valued functions defined on a set $E \subseteq \mathbb{C}$, and converging uniformly on E to a function f. Prove that f is continuous on E.

State the Weierstrass M-test for uniform convergence of a series $\sum_{n=1}^{\infty} u_n(z)$ of complex-valued functions on a set E.

Now let $f(z) = \sum_{n=1}^{\infty} u_n(z)$, where

$$u_n(z) = n^{-2} \sec\left(\pi z / 2n\right) .$$

Prove carefully that f is continuous on $\mathbb{C} \setminus \mathbb{Z}$.

[*You may assume the inequality* $|\cos z| \geqslant |\cos(\operatorname{Re} z)|$.]

11D Methods

The velocity potential $\phi(r, \theta)$ for inviscid flow in two dimensions satisfies the Laplace equation

$$\Delta \phi = \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \phi(r, \theta) = 0 \ .$$

(a) Using separation of variables, derive the general solution to the equation above that is single-valued and finite in each of the domains (i) $0 \leqslant r \leqslant a$; (ii) $a \leqslant r < \infty$.

(b) Assuming ϕ is single-valued, solve the Laplace equation subject to the boundary conditions $\frac{\partial \phi}{\partial r} = 0$ at $r = a$, and $\frac{\partial \phi}{\partial r} \to U \cos \theta$ as $r \to \infty$. Sketch the lines of constant potential.

12H Statistics

State and prove the Rao–Blackwell theorem.

Suppose that $X_1, \ldots, X_n$ are independent random variables uniformly distributed over $(\theta, 3\theta)$. Find a two-dimensional sufficient statistic $T(X)$ for θ. Show that an unbiased estimator of θ is $\hat{\theta} = X_1/2$.

Find an unbiased estimator of θ which is a function of $T(X)$ and whose mean square error is no more than that of $\hat{\theta}$.

13E Further Analysis

(a) State the residue theorem and use it to deduce the principle of the argument, in a form that involves winding numbers.

(b) Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\mathrm{Im}\,(p(z)) = 0$. Calculate $\mathrm{Re}\,(p(z))$ for each such z. [*It will be helpful to set* $z = e^{i\theta}$. *You may use the addition formulae* $\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ *and* $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$.]

(c) Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed path $\theta \mapsto e^{i\theta}$. Use your answer to (b) to give a rough sketch of the path $p \circ \gamma$, paying particular attention to where it crosses the real axis.

(d) Hence, or otherwise, determine for every real t the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = t$. (You need not give rigorous justifications for your calculations.)

14H Optimization

Use the two-phase simplex method to solve the problem

$$\begin{array}{lrcrcrcr}
\text{minimize} & 5x_1 & - & 12x_2 & + & 13x_3 & & \\[1em]
\text{subject to} & 4x_1 & + & 5x_2 & & & \leq & 9, \\
 & 6x_1 & + & 4x_2 & + & x_3 & \geq & 12, \\
 & 3x_1 & + & 2x_2 & - & x_3 & \leq & 3, \\[0.5em]
 & \multicolumn{7}{l}{x_i \geq 0, \qquad i = 1, 2, 3.}
\end{array}$$

15G Linear Mathematics

Let $\alpha \in L(U,V)$ be a linear map between finite-dimensional vector spaces. Let

$$M^l(\alpha) = \{\beta \in L(V,U) : \ \beta\,\alpha = 0\} \qquad \text{and}$$

$$M^r(\alpha) = \{\beta \in L(V,U) : \ \alpha\,\beta = 0\} \ .$$

(a) Prove that $M^l(\alpha)$ and $M^r(\alpha)$ are subspaces of $L(V,U)$ of dimensions

$$\dim M^l(\alpha) = (\dim V - \operatorname{rank}\alpha)\dim U \qquad \text{and}$$

$$\dim M^r(\alpha) = \dim\ker(\alpha)\dim V \ .$$

[*You may use the result that there exist bases in U and V so that α is represented by*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and r is the rank of α.]

(b) Let $\Phi\colon L(U,V) \to L(V^*,U^*)$ be given by $\Phi(\alpha) = \alpha^*$, where α^* is the dual map induced by α. Prove that Φ is an isomorphism. [*You may assume that Φ is linear, and you may use the result that a finite-dimensional vector space and its dual have the same dimension.*]

(c) Prove that

$$\Phi(M^l(\alpha)) = M^r(\alpha^*) \quad \text{and} \quad \Phi(M^r(\alpha)) = M^l(\alpha^*).$$

[*You may use the results that $(\beta\,\alpha)^* = \alpha^*\,\beta^*$ and that β^{**} can be identified with β under the canonical isomorphism between a vector space and its double dual.*]

(d) Conclude that $\operatorname{rank}(\alpha) = \operatorname{rank}(\alpha^*)$.

16C Fluid Dynamics

Define the terms *irrotational flow* and *incompressible flow*. The two-dimensional flow of an incompressible fluid is given in terms of a streamfunction $\psi(x,y)$ as

$$\mathbf{u} = (u,v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$$

in Cartesian coordinates (x,y). Show that the line integral

$$\int_{\mathbf{x_1}}^{\mathbf{x_2}} \mathbf{u}\cdot\mathbf{n}\, dl = \psi(\mathbf{x_2}) - \psi(\mathbf{x_1})$$

along any path joining the points $\mathbf{x_1}$ and $\mathbf{x_2}$, where $\mathbf{n}$ is the unit normal to the path. Describe how this result is related to the concept of mass conservation.

Inviscid, incompressible fluid is contained in the semi-infinite channel $x > 0$, $0 < y < 1$, which has rigid walls at $x = 0$ and at $y = 0,\ 1$, apart from a small opening at the origin through which the fluid is withdrawn with volume flux m per unit distance in the third dimension. Show that the streamfunction for irrotational flow in the channel can be chosen (up to an additive constant) to satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

and boundary conditions

$$\begin{aligned} &\psi = 0 \qquad \text{on } y = 0,\ x > 0, \\ &\psi = -m \qquad \text{on } x = 0,\ 0 < y < 1, \\ &\psi = -m \qquad \text{on } y = 1,\ x > 0, \\ &\psi \to -my \qquad \text{as } x \to \infty, \end{aligned}$$

if it is assumed that the flow at infinity is uniform. Solve the boundary-value problem above using separation of variables to obtain

$$\psi = -my + \frac{2m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\ \sin n\pi y\ e^{-n\pi x}.$$

17B Complex Methods

Let

$$f(z)=\frac{z^m}{1+z^n}, \quad n>m+1, \quad m,n\in\mathbb{N},$$

and let C_R be the boundary of the domain

$$D_R=\{z=re^{i\theta}:\ 0<r<R, \quad 0<\theta<\tfrac{2\pi}{n}\}, \quad R>1.$$

(a) Using the residue theorem, determine

$$\int_{C_R} f(z)\,dz.$$

(b) Show that the integral of $f(z)$ along the circular part γ_R of C_R tends to 0 as $R\to\infty$.

(c) Deduce that

$$\int_0^\infty \frac{x^m}{1+x^n}\,dx=\frac{\pi}{n\sin\frac{\pi(m+1)}{n}}\,.$$

18A Special Relativity

A pion of rest mass M decays at rest into a muon of rest mass $m<M$ and a neutrino of zero rest mass. What is the speed u of the muon?

In the pion rest frame S, the muon moves in the y-direction. A moving observer, in a frame S' with axes parallel to those in the pion rest frame, wishes to take measurements of the decay along the x-axis, and notes that the pion has speed v with respect to the x-axis. Write down the four-dimensional Lorentz transformation relating S' to S and determine the momentum of the muon in S'. Hence show that in S' the direction of motion of the muon makes an angle θ with respect to the y-axis, where

$$\tan\theta=\frac{M^2+m^2}{M^2-m^2}\frac{v}{(c^2-v^2)^{1/2}}\,.$$