UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 5 June 2003 9 to 12

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** . . . , **H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Analysis II

Let V be the vector space of continuous real-valued functions on $[-1,1]$. Show that the function

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx$$

defines a norm on V.

Let $f_n(x) = x^n$. Show that (f_n) is a Cauchy sequence in V. Is (f_n) convergent? Justify your answer.

2D Methods

Consider the path between two arbitrary points on a cone of interior angle 2α. Show that the arc-length of the path $r(\theta)$ is given by

$$\int (r^2 + r'^2 \operatorname{cosec}^2 \alpha)^{1/2}\, d\theta\ ,$$

where $r' = \frac{dr}{d\theta}$. By minimizing the total arc-length between the points, determine the equation for the shortest path connecting them.

3E Further Analysis

(a) Let $f \colon \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leqslant 1 + |z|^{1/2}$ for every z. Prove that f is constant.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\operatorname{Re}(f(z)) \geqslant 0$ for every z. Prove that f is constant.

4F Geometry

Show that any isometry of Euclidean 3-space which fixes the origin can be written as a composite of at most three reflections in planes through the origin, and give (with justification) an example of an isometry for which three reflections are necessary.

5H Optimization

Two players A and B play a zero-sum game with the pay-off matrix

	B_1	B_2	B_3
A_1	4	−2	−5
A_2	−2	4	3
A_3	−3	6	2
A_4	3	−8	−6

Here, the (i, j) entry of the matrix indicates the pay-off to player A if he chooses move A_i and player B chooses move B_j. Show that the game can be reduced to a zero-sum game with 2×2 pay-off matrix.

Determine the value of the game and the optimal strategy for player A.

6B Numerical Analysis

Given $(n + 1)$ distinct points $x_0, x_1, \ldots, x_n$, let

$$\ell_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - x_k}{x_i - x_k}$$

be the fundamental Lagrange polynomials of degree n, let

$$\omega(x) = \prod_{i=0}^{n} (x - x_i),$$

and let p be any polynomial of degree $\leq n$.

(a) Prove that $\sum_{i=0}^{n} p(x_i)\ell_i(x) \equiv p(x)$.

(b) Hence or otherwise derive the formula

$$\frac{p(x)}{\omega(x)} = \sum_{i=0}^{n} \frac{A_i}{x - x_i}, \quad A_i = \frac{p(x_i)}{\omega'(x_i)},$$

which is the decomposition of $p(x)/\omega(x)$ into partial fractions.

7G Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U and let β be another endomorphism of U that commutes with α. If λ is an eigenvalue of α, show that β maps the kernel of $\alpha - \lambda\,\iota$ into itself, where ι is the identity map. Suppose now that α is diagonalizable with n distinct real eigenvalues where $n = \dim U$. Prove that if there exists an endomorphism β of U such that $\alpha = \beta^2$, then $\lambda \geqslant 0$ for all eigenvalues λ of α.

8C Fluid Dynamics

Show that the velocity field

$$\mathbf{u} = \mathbf{U} + \frac{\mathbf{\Gamma} \times \mathbf{r}}{2\pi r^2},$$

where $\mathbf{U} = (U, 0, 0)$, $\mathbf{\Gamma} = (0, 0, \Gamma)$ and $\mathbf{r} = (x, y, 0)$ in Cartesian coordinates (x, y, z), represents the combination of a uniform flow and the flow due to a line vortex. Define and evaluate the circulation of the vortex.

Show that

$$\oint_{C_R} (\mathbf{u} \cdot \mathbf{n})\mathbf{u}\, dl \to \tfrac{1}{2}\mathbf{\Gamma} \times \mathbf{U} \qquad \text{as} \qquad R \to \infty,$$

where C_R is a circle $x^2 + y^2 = R^2$, $z = \text{const}$. Explain how this result is related to the lift force on a two-dimensional aerofoil or other obstacle.

9G Quadratic Mathematics

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Explain what is meant by the *discriminant* d of f. State a necessary and sufficient condition for some form of discriminant d to represent an odd prime number p. Using this result or otherwise, find the primes p which can be represented by the form $x^2 + 3y^2$.

10A Special Relativity

What are the momentum and energy of a photon of wavelength λ?

A photon of wavelength λ is incident on an electron. After the collision, the photon has wavelength λ'. Show that

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta),$$

where θ is the scattering angle and m is the electron rest mass.

SECTION II

11F Analysis II

State and prove the Contraction Mapping Theorem.

Let (X,d) be a bounded metric space, and let F denote the set of all continuous maps $X \to X$. Let $\rho\colon F \times F \to \mathbb{R}$ be the function

$$\rho(f,g) = \sup\{d(f(x),g(x)) : x \in X\} \ .$$

Show that ρ is a metric on F, and that (F,ρ) is complete if (X,d) is complete. [*You may assume that a uniform limit of continuous functions is continuous.*]

Now suppose that (X,d) is complete. Let $C \subseteq F$ be the set of contraction mappings, and let $\theta\colon C \to X$ be the function which sends a contraction mapping to its unique fixed point. Show that θ is continuous. [*Hint: fix $f \in C$ and consider $d(\theta(g), f(\theta(g)))$, where $g \in C$ is close to f.*]

12D Methods

The transverse displacement $y(x,t)$ of a stretched string clamped at its ends $x = 0, l$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2k\frac{\partial y}{\partial t} \ , \quad y(x,0) = 0, \ \frac{\partial y}{\partial t}(x,0) = \delta(x-a) \, ,$$

where $c > 0$ is the wave velocity, and $k > 0$ is the damping coefficient. The initial conditions correspond to a sharp blow at $x = a$ at time $t = 0$.

(a) Show that the subsequent motion of the string is given by

$$y(x,t) = \frac{1}{\sqrt{\alpha_n^2 - k^2}} \sum_n 2e^{-kt} \sin\frac{\alpha_n a}{c} \ \sin\frac{\alpha_n x}{c} \ \sin/(\sqrt{\alpha_n^2 - k^2} \ \ t)$$

where $\alpha_n = \pi c n / l$.

(b) Describe what happens in the limits of small and large damping. What critical parameter separates the two cases?

13E Further Analysis

(a) State Taylor's Theorem.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ be defined whenever $|z-z_0| < r$. Suppose that $z_k \to z_0$ as $k \to \infty$, that no z_k equals z_0 and that $f(z_k) = g(z_k)$ for every k. Prove that $a_n = b_n$ for every $n \geqslant 0$.

(c) Let D be a domain, let $z_0 \in D$ and let (z_k) be a sequence of points in D that converges to z_0, but such that no z_k equals z_0. Let $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ be analytic functions such that $f(z_k) = g(z_k)$ for every k. Prove that $f(z) = g(z)$ for every $z \in D$.

(d) Let D be the domain $\mathbb{C}\setminus\{0\}$. Give an example of an analytic function $f\colon D \to \mathbb{C}$ such that $f(n^{-1}) = 0$ for every positive integer n but f is not identically 0.

(e) Show that any function with the property described in (d) must have an essential singularity at the origin.

14F Geometry

State and prove the Gauss–Bonnet formula for the area of a spherical triangle. Deduce a formula for the area of a spherical n-gon with angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. For what range of values of α does there exist a (convex) regular spherical n-gon with angle α?

Let Δ be a spherical triangle with angles π/p, π/q and π/r where p, q, r are integers, and let G be the group of isometries of the sphere generated by reflections in the three sides of Δ. List the possible values of (p, q, r), and in each case calculate the order of the corresponding group G. If $(p, q, r) = (2, 3, 5)$, show how to construct a regular dodecahedron whose group of symmetries is G.

[*You may assume that the images of Δ under the elements of G form a tessellation of the sphere.*]

15H Optimization

Explain what is meant by a transportation problem where the total demand equals the total supply. Write the Lagrangian and describe an algorithm for solving such a problem. Starting from the north-west initial assignment, solve the problem with three sources and three destinations described by the table

5	9	1	36
3	10	6	84
7	2	5	40
14	68	78	

where the figures in the 3×3 box denote the transportation costs (per unit), the right-hand column denotes supplies, and the bottom row demands.

16B Numerical Analysis

The functions $H_0, H_1, \ldots$ are generated by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \,.$$

(a) Show that H_n is a polynomial of degree n, and that the H_n are orthogonal with respect to the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2} \, dx \,.$$

(b) By induction or otherwise, prove that the H_n satisfy the three-term recurrence relation

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \,.$$

[*Hint: you may need to prove the equality* $H_n'(x) = 2n H_{n-1}(x)$ *as well.*]

17G Linear Mathematics

Define the *determinant* $\det(A)$ of an $n \times n$ complex matrix A. Let $A_1, \ldots, A_n$ be the columns of A, let σ be a permutation of $\{1, \ldots, n\}$ and let A^σ be the matrix whose columns are $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$. Prove from your definition of determinant that $\det(A^\sigma) = \epsilon(\sigma) \det(A)$, where $\epsilon(\sigma)$ is the sign of the permutation σ. Prove also that $\det(A) = \det(A^t)$.

Define the *adjugate* matrix $\operatorname{adj}(A)$ and prove from your definitions that $A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) I$, where I is the identity matrix. Hence or otherwise, prove that if $\det(A) \neq 0$, then A is invertible.

Let C and D be real $n \times n$ matrices such that the complex matrix $C + iD$ is invertible. By considering $\det(C + \lambda D)$ as a function of λ or otherwise, prove that there exists a real number λ such that $C + \lambda D$ is invertible. [*You may assume that if a matrix A is invertible, then* $\det(A) \neq 0$.]

Deduce that if two real matrices A and B are such that there exists an invertible complex matrix P with $P^{-1} A P = B$, then there exists an invertible **real** matrix Q such that $Q^{-1} A Q = B$.

18C Fluid Dynamics

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid in the absence of gravity.

Water of density ρ is driven through a tube of length L and internal radius a by the pressure exerted by a spherical, water-filled balloon of radius $R(t)$ attached to one end of the tube. The balloon maintains the pressure of the water entering the tube at $2\gamma/R$ in excess of atmospheric pressure, where γ is a constant. It may be assumed that the water exits the tube at atmospheric pressure. Show that

$$R^3\ddot{R} + 2R^2\dot{R}^2 = -\frac{\gamma a^2}{2\rho L}. \qquad (\dagger)$$

Solve equation ($\dagger$), by multiplying through by $2R\dot{R}$ or otherwise, to obtain

$$t = R_0^2\left(\frac{2\rho L}{\gamma a^2}\right)^{1/2}\left[\frac{\pi}{4} - \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right],$$

where $\theta = \sin^{-1}(R/R_0)$ and R_0 is the initial radius of the balloon. Hence find the time when $R = 0$.

19G Quadratic Mathematics

Let U be a finite-dimensional real vector space endowed with a positive definite inner product. A linear map $\tau : U \to U$ is said to be an *orthogonal projection* if τ is self-adjoint and $\tau^2 = \tau$.

(a) Prove that for every orthogonal projection τ there is an orthogonal decomposition

$$U = \ker(\tau) \oplus \operatorname{im}(\tau).$$

(b) Let $\phi\colon U \to U$ be a linear map. Show that if $\phi^2 = \phi$ and $\phi\,\phi^* = \phi^*\,\phi$, where ϕ^* is the adjoint of ϕ, then ϕ is an orthogonal projection. [*You may find it useful to prove first that if $\phi\,\phi^* = \phi^*\,\phi$, then ϕ and ϕ^* have the same kernel.*]

(c) Show that given a subspace W of U there exists a unique orthogonal projection τ such that $\operatorname{im}(\tau) = W$. If W_1 and W_2 are two subspaces with corresponding orthogonal projections τ_1 and τ_2, show that $\tau_2 \circ \tau_1 = 0$ if and only if W_1 is orthogonal to W_2.

(d) Let $\phi\colon U \to U$ be a linear map satisfying $\phi^2 = \phi$. Prove that one can define a positive definite inner product on U such that ϕ becomes an orthogonal projection.

20A Quantum Mechanics

The radial wavefunction for the hydrogen atom satisfies the equation

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right)+\frac{\hbar^2}{2mr^2}\ell(\ell+1)R(r)-\frac{e^2}{4\pi\epsilon_0 r}R(r)=ER(r)\,.$$

Explain the origin of each term in this equation.

The wavefunctions for the ground state and first radially excited state, both with $\ell = 0$, can be written as

$$R_1(r) = N_1 \exp(-\alpha r)$$
$$R_2(r) = N_2(r+b)\exp(-\beta r)$$

respectively, where N_1 and N_2 are normalization constants. Determine α, β, b and the corresponding energy eigenvalues E_1 and E_2.

A hydrogen atom is in the first radially excited state. It makes the transition to the ground state, emitting a photon. What is the frequency of the emitted photon?