UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 4 June 2003 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B, . . . , H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Analysis II

Explain what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ to be *differentiable* at a point (a, b). Show that if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in a neighbourhood of (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Let

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad ((x, y) \neq (0, 0))$$

and $f(0, 0) = 0$. Do the partial derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$? Justify your answers.

2C Methods

Explain briefly why the second-rank tensor

$$\int_S x_i x_j \, dS(\mathbf{x})$$

is isotropic, where S is the surface of the unit sphere centred on the origin.

A second-rank tensor is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j) \, dS(\mathbf{x}) \, ,$$

where S is the surface of the unit sphere centred on the origin. Calculate $T(\mathbf{y})$ in the form

$$T_{ij} = \lambda \delta_{ij} + \mu y_i y_j \, ,$$

where λ and μ are to be determined.

By considering the action of T on $\mathbf{y}$ and on vectors perpendicular to $\mathbf{y}$, determine the eigenvalues and associated eigenvectors of T.

3H Statistics

Let $X_1, \ldots, X_n$ be a random sample from the $N(\theta, \sigma^2)$ distribution, and suppose that the prior distribution for θ is $N(\mu, \tau^2)$, where σ^2, μ, τ^2 are known. Determine the posterior distribution for θ, given $X_1, \ldots, X_n$, and the best point estimate of θ under both quadratic and absolute error loss.

4E Further Analysis

Let τ_1 be the collection of all subsets $A \subset \mathbb{N}$ such that $A = \emptyset$ or $\mathbb{N} \setminus A$ is finite. Let τ_2 be the collection of all subsets of $\mathbb{N}$ of the form $I_n = \{n, n+1, n+2, \ldots\}$, together with the empty set. Prove that τ_1 and τ_2 are both topologies on $\mathbb{N}$.

Show that a function f from the topological space $(\mathbb{N}, \tau_1)$ to the topological space $(\mathbb{N}, \tau_2)$ is continuous if and only if one of the following alternatives holds:

(i) $f(n) \to \infty$ as $n \to \infty$;

(ii) there exists $N \in \mathbb{N}$ such that $f(n) = N$ for all but finitely many n and $f(n) \leqslant N$ for all n.

5B Numerical Analysis

Let

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} \gamma \\ 0 \\ 0 \\ \gamma a \end{pmatrix}, \quad \gamma = 1 - a^4 \neq 0.$$

Find the LU factorization of the matrix A and use it to solve the system $Ax = b$.

6E Linear Mathematics

Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. For each i let $\mathbf{v}_i$ be the vector $(1, a_i, a_i^2, \ldots, a_i^{n-1})$. Let A be the $n \times n$ matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and let $\mathbf{c}$ be a column vector of size n. Prove that $A\mathbf{c} = \mathbf{0}$ if and only if $\mathbf{c} = \mathbf{0}$. Deduce that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$.

[*You may use general facts about matrices if you state them clearly.*]

7B Complex Methods

(a) Using the residue theorem, evaluate

$$\int_{|z|=1} \left(z - \frac{1}{z}\right)^{2n} \frac{dz}{z}\,.$$

(b) Deduce that

$$\int_0^{2\pi} \sin^{2n} t\, dt = \frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}\,.$$

8G Quadratic Mathematics

Let U be a finite-dimensional real vector space and b a positive definite symmetric bilinear form on $U \times U$. Let $\psi : U \to U$ be a linear map such that $b(\psi(x), y) + b(x, \psi(y)) = 0$ for all x and y in U. Prove that if ψ is invertible, then the dimension of U must be even. By considering the restriction of ψ to its image or otherwise, prove that the rank of ψ is always even.

9A Quantum Mechanics

What is meant by the statement than an operator is *hermitian*?

A particle of mass m moves in the real potential $V(x)$ in one dimension. Show that the Hamiltonian of the system is hermitian.

Show that

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle \,,$$
$$\frac{d}{dt}\langle p\rangle = \langle -V'(x)\rangle \,,$$

where p is the momentum operator and $\langle A\rangle$ denotes the expectation value of the operator A.

SECTION II

10F Analysis II

Let V be the space of $n \times n$ real matrices. Show that the function

$$N(A) = \sup \{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| = 1\}$$

(where $\| - \|$ denotes the usual Euclidean norm on $\mathbb{R}^n$) defines a norm on V. Show also that this norm satisfies $N(AB) \leqslant N(A)N(B)$ for all A and B, and that if $N(A) < \epsilon$ then all entries of A have absolute value less than ϵ. Deduce that any function $f : V \to \mathbb{R}$ such that $f(A)$ is a polynomial in the entries of A is continuously differentiable.

Now let $d : V \to \mathbb{R}$ be the mapping sending a matrix to its determinant. By considering $d(I + H)$ as a polynomial in the entries of H, show that the derivative $d'(I)$ is the function $H \mapsto \operatorname{tr} H$. Deduce that, for any A, $d'(A)$ is the mapping $H \mapsto \operatorname{tr}((\operatorname{adj} A)H)$, where $\operatorname{adj} A$ is the adjugate of A, i.e. the matrix of its cofactors.

[*Hint: consider first the case when A is invertible. You may assume the results that the set U of invertible matrices is open in V and that its closure is the whole of V, and the identity* $(\operatorname{adj} A)A = \det A.I$.]

11C Methods

State the transformation law for an nth-rank tensor $T_{ij\cdots k}$.

Show that the fourth-rank tensor

$$c_{ijkl} = \alpha\, \delta_{ij}\, \delta_{kl} + \beta\, \delta_{ik}\, \delta_{jl} + \gamma\, \delta_{il}\, \delta_{jk}$$

is isotropic for arbitrary scalars α, β and γ.

The stress σ_{ij} and strain e_{ij} in a linear elastic medium are related by

$$\sigma_{ij} = c_{ijkl}\, e_{kl}.$$

Given that e_{ij} is symmetric and that the medium is isotropic, show that the stress-strain relationship can be written in the form

$$\sigma_{ij} = \lambda\, e_{kk}\, \delta_{ij} + 2\mu\, e_{ij}.$$

Show that e_{ij} can be written in the form $e_{ij} = p\delta_{ij} + d_{ij}$, where d_{ij} is a traceless tensor and p is a scalar to be determined. Show also that necessary and sufficient conditions for the stored elastic energy density $E = \frac{1}{2}\sigma_{ij}\, e_{ij}$ to be non-negative for any deformation of the solid are that

$$\mu \geq 0 \qquad \text{and} \qquad \lambda \geq -\tfrac{2}{3}\mu.$$

12H Statistics

An examination was given to 500 high-school students in each of two large cities, and their grades were recorded as low, medium, or high. The results are given in the table below.

	Low	*Medium*	*High*
City A	103	145	252
City B	140	136	224

Derive carefully the test of homogeneity and test the hypothesis that the distributions of scores among students in the two cities are the same.

[*Hint:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_5^2	χ_6^2
99% percentile	6.63	9.21	11.34	15.09	16.81
95% percentile	3.84	5.99	7.81	11.07	12.59

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13E Further Analysis

(a) Let $f\colon [1,\infty) \to \mathbb{C}$ be defined by $f(t) = t^{-1}e^{2\pi i t}$ and let X be the image of f. Prove that $X \cup \{0\}$ is compact and path-connected. [*Hint: you may find it helpful to set* $s = t^{-1}$.]

(b) Let $g\colon [1,\infty) \to \mathbb{C}$ be defined by $g(t) = (1+t^{-1})e^{2\pi i t}$, let Y be the image of g and let $\overline{D}$ be the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Prove that $Y \cup \overline{D}$ is connected. Explain briefly why it is not path-connected.

14B Numerical Analysis

Let

$$f''(0) \approx a_0 f(-1) + a_1 f(0) + a_2 f(1) = \mu(f)$$

be an approximation of the second derivative which is exact for $f \in \mathcal{P}_2$, the set of polynomials of degree ≤ 2, and let

$$e(f) = f''(0) - \mu(f)$$

be its error.

(a) Determine the coefficients a_0, a_1, a_2.

(b) Using the Peano kernel theorem prove that, for $f \in C^3[-1,1]$, the set of three-times continuously differentiable functions, the error satisfies the inequality

$$|e(f)| \leq \frac{1}{3} \max_{x\in[-1,1]} |f'''(x)|\,.$$

15E Linear Mathematics

(a) Let $A = (a_{ij})$ be an $m \times n$ matrix and for each $k \leqslant n$ let A_k be the $m \times k$ matrix formed by the first k columns of A. Suppose that $n > m$. Explain why the nullity of A is non-zero. Prove that if k is minimal such that A_k has non-zero nullity, then the nullity of A_k is 1.

(b) Suppose that no column of A consists entirely of zeros. Deduce from (a) that there exist scalars $b_1, \ldots, b_k$ (where k is defined as in (a)) such that $\sum_{j=1}^{k} a_{ij} b_j = 0$ for every $i \leqslant m$, but whenever $\lambda_1, \ldots, \lambda_k$ are distinct real numbers there is some $i \leqslant m$ such that $\sum_{j=1}^{k} a_{ij} \lambda_j b_j \neq 0$.

(c) Now let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be bases for the same real m-dimensional vector space. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real numbers such that for every j the vectors $\mathbf{v}_1 + \lambda_j \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda_j \mathbf{w}_m$ are linearly dependent. For each j, let $a_{1j}, \ldots, a_{mj}$ be scalars, not all zero, such that $\sum_{i=1}^{m} a_{ij}(\mathbf{v}_i + \lambda_j \mathbf{w}_i) = \mathbf{0}$. By applying the result of (b) to the matrix (a_{ij}), deduce that $n \leqslant m$.

(d) It follows that the vectors $\mathbf{v}_1 + \lambda \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda \mathbf{w}_m$ are linearly dependent for at most m values of λ. Explain briefly how this result can also be proved using determinants.

16B Complex Methods

(a) Show that if f satisfies the equation

$$f''(x) - x^2 f(x) = \mu f(x), \quad x \in \mathbb{R}, \qquad (*)$$

where μ is a constant, then its Fourier transform $\widehat{f}$ satisfies the same equation, i.e.

$$\widehat{f}''(\lambda) - \lambda^2 \widehat{f}(\lambda) = \mu \widehat{f}(\lambda) .$$

(b) Prove that, for each $n \geq 0$, there is a polynomial $p_n(x)$ of degree n, unique up to multiplication by a constant, such that

$$f_n(x) = p_n(x) e^{-x^2/2}$$

is a solution of $(*)$ for some $\mu = \mu_n$.

(c) Using the fact that $g(x) = e^{-x^2/2}$ satisfies $\widehat{g} = cg$ for some constant c, show that the Fourier transform of f_n has the form

$$\widehat{f_n}(\lambda) = q_n(\lambda) e^{-\lambda^2/2}$$

where q_n is also a polynomial of degree n.

(d) Deduce that the f_n are eigenfunctions of the Fourier transform operator, i.e. $\widehat{f_n}(x) = c_n f_n(x)$ for some constants c_n.

17G Quadratic Mathematics

Let S be the set of all 2×2 complex matrices A which are *hermitian*, that is, $A^* = A$, where $A^* = \overline{A}^t$.

(a) Show that S is a real 4-dimensional vector space. Consider the real symmetric bilinear form b on this space defined by

$$b(A, B) = \tfrac{1}{2}\left(\operatorname{tr}(A\,B) - \operatorname{tr}(A)\operatorname{tr}(B)\right) .$$

Prove that $b(A, A) = -\det A$ and $b(A, I) = -\frac{1}{2}\operatorname{tr}(A)$, where I denotes the identity matrix.

(b) Consider the three matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Prove that the basis I, A_1, A_2, A_3 of S diagonalizes b. Hence or otherwise find the rank and signature of b.

(c) Let Q be the set of all 2×2 complex matrices C which satisfy $C + C^* = \operatorname{tr}(C)\,I$. Show that Q is a real 4-dimensional vector space. Given $C \in Q$, put

$$\Phi(C) = \frac{1-i}{2}\operatorname{tr}(C)\,I + i\,C.$$

Show that Φ takes values in S and is a linear isomorphism between Q and S.

(d) Define a real symmetric bilinear form on Q by setting $c(C, D) = -\frac{1}{2}\operatorname{tr}(C\,D)$, $C, D \in Q$. Show that $b(\Phi(C), \Phi(D)) = c(C, D)$ for all $C, D \in Q$. Find the rank and signature of the symmetric bilinear form c defined on Q.

18A Quantum Mechanics

A particle of mass m and energy E moving in one dimension is incident from the left on a potential barrier $V(x)$ given by

$$V(x) = \begin{cases} V_0 & 0 \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

with $V_0 > 0$.

In the limit $V_0 \to \infty, a \to 0$ with $V_0 a = U$ held fixed, show that the transmission probability is

$$T = \left(1 + \frac{mU^2}{2E\hbar^2}\right)^{-1} .$$