UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 3 June 2003 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked **A**, **B** *and* **E** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Algebra and Geometry

The mapping α of $\mathbb{R}^3$ into itself is a reflection in the plane $x_2 = x_3$. Find the matrix A of α with respect to any basis of your choice, which should be specified.

The mapping β of $\mathbb{R}^3$ into itself is a rotation about the line $x_1 = x_2 = x_3$ through $2\pi/3$, followed by a dilatation by a factor of 2. Find the matrix B of β with respect to a choice of basis that should again be specified.

Show explicitly that

$$B^3 = 8A^2$$

and explain why this must hold, irrespective of your choices of bases.

2B Algebra and Geometry

Show that if a group G contains a normal subgroup of order 3, and a normal subgroup of order 5, then G contains an element of order 15.

Give an example of a group of order 10 with no element of order 10.

3A Vector Calculus

Sketch the curve $y^2 = x^2 + 1$. By finding a parametric representation, or otherwise, determine the points on the curve where the radius of curvature is least, and compute its value there.

[*Hint: you may use the fact that the radius of curvature of a parametrized curve* $(x(t), y(t))$ *is* $(\dot{x}^2 + \dot{y}^2)^{3/2}/|\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.]

4A Vector Calculus

Suppose V is a region in $\mathbb{R}^3$, bounded by a piecewise smooth closed surface S, and $\phi(\mathbf{x})$ is a scalar field satisfying

$$\begin{aligned} \boldsymbol{\nabla}^2\phi &= 0 \quad \text{in } V, \\ \text{and} \quad \phi &= f(\mathbf{x}) \quad \text{on } S\,. \end{aligned}$$

Prove that ϕ is determined uniquely in V.

How does the situation change if the normal derivative of ϕ rather than ϕ itself is specified on S?

SECTION II

5E Algebra and Geometry

(a) Show, using vector methods, that the distances from the centroid of a tetrahedron to the centres of opposite pairs of edges are equal. If the three distances are u, v, w and if a, b, c, d are the distances from the centroid to the vertices, show that

$$u^2 + v^2 + w^2 = \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2).$$

[*The centroid of k points in $\mathbb{R}^3$ with position vectors $\mathbf{x}_i$ is the point with position vector* $\dfrac{1}{k}\sum \mathbf{x}_i$.]

(b) Show that

$$|\mathbf{x} - \mathbf{a}|^2 \cos^2 \alpha = [(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n}]^2,$$

with $\mathbf{n}^2 = 1$, is the equation of a right circular double cone whose vertex has position vector $\mathbf{a}$, axis of symmetry $\mathbf{n}$ and opening angle α.

Two such double cones, with vertices $\mathbf{a}_1$ and $\mathbf{a}_2$, have parallel axes and the same opening angle. Show that if $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 \neq \mathbf{0}$, then the intersection of the cones lies in a plane with unit normal

$$\mathbf{N} = \frac{\mathbf{b}\cos^2\alpha - \mathbf{n}(\mathbf{n}\cdot\mathbf{b})}{\sqrt{\mathbf{b}^2\cos^4\alpha + (\mathbf{b}\cdot\mathbf{n})^2(1 - 2\cos^2\alpha)}}\,.$$

6E Algebra and Geometry

Derive an expression for the triple scalar product $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3$ in terms of the determinant of the matrix E whose rows are given by the components of the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Use the geometrical interpretation of the cross product to show that $\mathbf{e}_a$, $a = 1, 2, 3$, will be a *not necessarily orthogonal* basis for $\mathbb{R}^3$ as long as $\det E \neq 0$.

The rows of another matrix $\hat{E}$ are given by the components of three other vectors $\hat{\mathbf{e}}_b$, $b = 1, 2, 3$. By considering the matrix $E\hat{E}^{\mathrm{T}}$, where ${}^{\mathrm{T}}$ denotes the transpose, show that there is a unique choice of $\hat{E}$ such that $\hat{\mathbf{e}}_b$ is also a basis and

$$\mathbf{e}_a \cdot \hat{\mathbf{e}}_b = \delta_{ab}.$$

Show that the new basis is given by

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3} \qquad \text{etc.}$$

Show that if either $\mathbf{e}_a$ or $\hat{\mathbf{e}}_b$ is an orthonormal basis then E is a rotation matrix.

7B Algebra and Geometry

Let G be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $X = \{\alpha, \beta, \gamma\}$ be a set of three distinct points in $\mathbb{C} \cup \{\infty\}$.

(i) Show that there exists a $g \in G$ sending α to 0, β to 1, and γ to ∞.

(ii) Hence show that if $H = \{g \in G \mid gX = X\}$, then H is isomorphic to S_3, the symmetric group on 3 letters.

8B Algebra and Geometry

(a) Determine the characteristic polynomial and the eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

Is it diagonalizable?

(b) Show that an $n \times n$ matrix A with characteristic polynomial $f(t) = (t - \mu)^n$ is diagonalizable if and only if $A = \mu I$.

9A Vector Calculus

Let C be the closed curve that is the boundary of the triangle T with vertices at the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Specify a direction along C and consider the integral

$$\oint_C \mathbf{A} \cdot d\mathbf{x} \ ,$$

where $\mathbf{A} = (z^2 - y^2,\ x^2 - z^2,\ y^2 - x^2)$. Explain why the contribution to the integral is the same from each edge of C, and evaluate the integral.

State Stokes's theorem and use it to evaluate the surface integral

$$\int_T (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S} \ ,$$

the components of the normal to T being positive.

Show that $d\mathbf{S}$ in the above surface integral can be written in the form $(1,1,1)\,dy\,dz$. Use this to verify your result by a direct calculation of the surface integral.

10A Vector Calculus

Write down an expression for the Jacobian J of a transformation

$$(x,y,z) \to (u,v,w)\,.$$

Use it to show that

$$\int_D f \ dx\,dy\,dz = \int_\Delta \phi \ |J| \ du\,dv\,dw$$

where D is mapped one-to-one onto Δ, and

$$\phi(u,v,w) \ = \ f\big(x(u,v,w),\ y(u,v,w),\ z(u,v,w)\big)\,.$$

Find a transformation that maps the ellipsoid D,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\ ,$$

onto a sphere. Hence evaluate

$$\int_D x^2\,dx\,dy\,dz\,.$$

11A Vector Calculus

(a) Prove the identity

$$\nabla(\mathbf{F} \cdot \mathbf{G}) \;=\; (\mathbf{F} \cdot \nabla)\mathbf{G} \;+\; (\mathbf{G} \cdot \nabla)\mathbf{F} \;+\; \mathbf{F} \times (\nabla \times \mathbf{G}) \;+\; \mathbf{G} \times (\nabla \times \mathbf{F})\,.$$

(b) If $\mathbf{E}$ is an irrotational vector field ($\nabla \times \mathbf{E} = \mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E} = -\nabla\phi$.

Show that

$$(2xy^2ze^{-x^2z},\ -2ye^{-x^2z},\ x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

12A Vector Calculus

State the divergence theorem. By applying this to $f(\mathbf{x})\mathbf{k}$, where $f(\mathbf{x})$ is a scalar field in a closed region V in $\mathbb{R}^3$ bounded by a piecewise smooth surface S, and $\mathbf{k}$ an arbitrary constant vector, show that

$$\int_V \nabla f \, dV = \int_S f \, d\mathbf{S}\,. \qquad (*)$$

A vector field $\mathbf{G}$ satisfies

$$\nabla \cdot \mathbf{G} = \rho(\mathbf{x})$$
$$\text{with} \quad \rho(\mathbf{x}) = \begin{cases} \rho_0 & |\mathbf{x}| \leqslant a \\ 0 & |\mathbf{x}| > a\,. \end{cases}$$

By applying the divergence theorem to $\int_V \nabla \cdot \mathbf{G} \, dV$, prove Gauss's law

$$\int_S \mathbf{G} \cdot d\mathbf{S} = \int_V \rho(\mathbf{x}) \, dV,$$

where S is the piecewise smooth surface bounding the volume V.

Consider the spherically symmetric solution

$$\mathbf{G}(\mathbf{x}) = G(r)\,\frac{\mathbf{x}}{r}\ ,$$

where $r = |\mathbf{x}|$. By using Gauss's law with S a sphere of radius r, centre $\mathbf{0}$, in the two cases $0 < r \leqslant a$ and $r > a$, show that

$$\mathbf{G}(\mathbf{x}) = \begin{cases} \dfrac{\rho_0}{3}\,\mathbf{x} & r \leqslant a \\ \dfrac{\rho_0}{3}\left(\dfrac{a}{r}\right)^3 \mathbf{x} & r > a\,. \end{cases}$$

The scalar field $f(\mathbf{x})$ satisfies $\mathbf{G} = \nabla f$. Assuming that $f \to 0$ as $r \to \infty$, and that f is continuous at $r = a$, find f everywhere.

By using a symmetry argument, explain why $(*)$ is clearly satisfied for this f if S is any sphere centred at the origin.