UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 29 May 2003 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked **A**, **B**, **C**, **D** *and* **F** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Algebra and Geometry

(a) Write the permutation

$$(123)(234)$$

as a product of disjoint cycles. Determine its order. Compute its sign.

(b) Elements x and y of a group G are *conjugate* if there exists a $g \in G$ such that $gxg^{-1} = y$.

Show that if permutations x and y are conjugate, then they have the same sign and the same order. Is the converse true? (Justify your answer with a proof or counter-example.)

2D Algebra and Geometry

Find the characteristic equation, the eigenvectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and the corresponding eigenvalues $\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}, \lambda_{\mathbf{d}}$ of the matrix

$$A = \begin{pmatrix} i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & i \end{pmatrix} .$$

Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ spans the complex vector space $\mathbb{C}^4$.

Consider the four subspaces of $\mathbb{C}^4$ defined parametrically by

$$\mathbf{z} = s\mathbf{a}, \quad \mathbf{z} = s\mathbf{b}, \quad \mathbf{z} = s\mathbf{c}, \quad \mathbf{z} = s\mathbf{d} \qquad (\mathbf{z} \in \mathbb{C}^4,\ s \in \mathbb{C}) .$$

Show that multiplication by A maps three of these subspaces onto themselves, and the remaining subspace into a smaller subspace to be specified.

3B Analysis

Define what it means for a function of a real variable to be *differentiable* at $x \in \mathbb{R}$.

Prove that if a function is differentiable at $x \in \mathbb{R}$, then it is continuous there.

Show directly from the definition that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at 0 with derivative 0.

Show that the derivative $f'(x)$ is not continuous at 0.

4C Analysis

Explain what is meant by the *radius of convergence* of a power series.

Find the radius of convergence R of each of the following power series:

$$\text{(i)} \quad \sum_{n=1}^{\infty} n^{-2} z^n, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \left(n + \frac{1}{2^n} \right) z^n .$$

In each case, determine whether the series converges on the circle $|z| = R$.

SECTION II

5B Algebra and Geometry

(a) In the standard basis of $\mathbb{R}^2$, write down the matrix for a rotation through an angle θ about the origin.

(b) Let A be a real 3×3 matrix such that $\det A = 1$ and $AA^{\mathrm{T}} = I$, where A^{T} is the transpose of A.

(i) Suppose that A has an eigenvector $\mathbf{v}$ with eigenvalue 1. Show that A is a rotation through an angle θ about the line through the origin in the direction of $\mathbf{v}$, where $\cos\theta = \frac{1}{2}(\operatorname{trace} A - 1)$.

(ii) Show that A must have an eigenvector $\mathbf{v}$ with eigenvalue 1.

6A Algebra and Geometry

Let α be a linear map

$$\alpha : \mathbb{R}^3 \to \mathbb{R}^3 .$$

Define the kernel K and image I of α.

Let $\mathbf{y} \in \mathbb{R}^3$. Show that the equation $\alpha\mathbf{x} = \mathbf{y}$ has a solution $\mathbf{x} \in \mathbb{R}^3$ if and only if $\mathbf{y} \in I$.

Let α have the matrix

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t & -2b \\ 1 & t & 0 \end{pmatrix}$$

with respect to the standard basis, where $b \in \mathbb{R}$ and t is a real variable. Find K and I for α. Hence, or by evaluating the determinant, show that if $0 < b < 2$ and $\mathbf{y} \in I$ then the equation $\alpha\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x} \in \mathbb{R}^3$ for all values of t.

7B Algebra and Geometry

(i) State the orbit-stabilizer theorem for a group G acting on a set X.

(ii) Denote the group of *all* symmetries of the cube by G. Using the orbit-stabilizer theorem, show that G has 48 elements.

Does G have any non-trivial normal subgroups?

Let L denote the line between two diagonally opposite vertices of the cube, and let

$$H = \{g \in G \mid gL = L\}$$

be the subgroup of symmetries that preserve the line. Show that H is isomorphic to the direct product $S_3 \times C_2$, where S_3 is the symmetric group on 3 letters and C_2 is the cyclic group of order 2.

8D Algebra and Geometry

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy–Schwarz inequality, and deduce the triangle inequality.

By means of a sketch, give a geometric interpretation of the scalar product $\mathbf{x} \cdot \mathbf{y}$ in the case $n = 3$, relating the value of $\mathbf{x} \cdot \mathbf{y}$ to the angle α between the directions of $\mathbf{x}$ and $\mathbf{y}$.

What is a unit vector? Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u}\,.$$

Show that

(i) for any fixed, linearly independent $\mathbf{u}$ and $\mathbf{v}$, the minimum of S over $\mathbf{w}$ is attained when $\mathbf{w} = \lambda(\mathbf{u} + \mathbf{v})$ for some $\lambda \in \mathbb{R}$;

(ii) $\lambda \leqslant -\frac{1}{2}$ in all cases;

(iii) $\lambda = -1$ and $S = -3/2$ in the case where $\mathbf{u} \cdot \mathbf{v} = \cos(2\pi/3)$.

9F Analysis

Prove the Axiom of Archimedes.

Let x be a real number in $[0, 1]$, and let m, n be positive integers. Show that the limit

$$\lim_{m\to\infty} \left[\lim_{n\to\infty} \cos^{2n}(m!\pi x) \right]$$

exists, and that its value depends on whether x is rational or irrational.

[*You may assume standard properties of the cosine function provided they are clearly stated.*]

10F Analysis

State without proof the *Integral Comparison Test* for the convergence of a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms.

Determine for which positive real numbers α the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges.

In each of the following cases determine whether the series is convergent or divergent:

(i) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n}$,

(ii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n \log\, n)\ (\log \log\, n)^2}$,

(iii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{(1+1/n)} \log n}$.

11B Analysis

Let $f : [a,b] \to \mathbb{R}$ be continuous. Define the *integral* $\int_a^b f(x)dx$. (You are not asked to prove existence.)

Suppose that m, M are real numbers such that $m \leqslant f(x) \leqslant M$ for all $x \in [a,b]$. Stating clearly any properties of the integral that you require, show that

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)\,.$$

The function $g : [a,b] \to \mathbb{R}$ is continuous and non-negative. Show that

$$m \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M \int_a^b g(x)dx\,.$$

Now let f be continuous on $[0,1]$. By suitable choice of g show that

$$\lim_{n\to\infty} \int_0^{1/\sqrt{n}} n f(x) e^{-nx} dx = f(0)\ ,$$

and by making an appropriate change of variable, or otherwise, show that

$$\lim_{n\to\infty} \int_0^1 n f(x) e^{-nx} dx = f(0)\,.$$

12C Analysis

State carefully the formula for integration by parts for functions of a real variable.

Let $f : (-1,1) \to \mathbb{R}$ be infinitely differentiable. Prove that for all $n \geqslant 1$ and all $t \in (-1,1)$,

$$f(t) = f(0) + f'(0)t + \frac{1}{2!}f''(0)t^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(0)t^{n-1} + \frac{1}{(n-1)!}\int_0^t f^{(n)}(x)(t-x)^{n-1}\,dx.$$

By considering the function $f(x) = \log(1-x)$ at $x = 1/2$, or otherwise, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

converges to $\log 2$.