UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Tuesday 4 June 2002 9 to 12

PAPER 2

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***1G, 17G*** *should be in one bundle and* ***13L, 15L*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1G Principles of Dynamics

(i) A number N of non-interacting particles move in one dimension in a potential $V(x,t)$. Write down the Hamiltonian and Hamilton's equations for one particle.

At time t, the number density of particles in phase space is $f(x,p,t)$. Write down the time derivative of f along a particle's trajectory. By equating the rate of change of the number of particles in a fixed domain V in phase space to the flux into V across its boundary, deduce that f is a constant along any particle's trajectory.

(ii) Suppose that $V(x) = \frac{1}{2}m\omega^2x^2$, and particles are injected in such a manner that the phase space density is a constant f_1 at any point of phase space corresponding to a particle energy being smaller than E_1 and zero elsewhere. How many particles are present?

Suppose now that the potential is very slowly altered to the square well form

$$V(x) = \begin{cases} 0, & -L < x < L \\ \infty & \text{elsewhere} \,. \end{cases}$$

Show that the greatest particle energy is now

$$E_2 = \frac{\pi^2}{8}\frac{E_1^2}{mL^2\omega^2}.$$

2K Functional Analysis

(i) State and prove the parallelogram law for Hilbert spaces.

Suppose that K is a closed linear subspace of a Hilbert space H and that $x \in H$. Show that x is orthogonal to K if and only if 0 is the nearest point to x in K.

(ii) Suppose that H is a Hilbert space and that ϕ is a continuous linear functional on H with $\|\phi\| = 1$. Show that there is a sequence (h_n) of unit vectors in H with $\phi(h_n)$ real and $\phi(h_n) > 1 - 1/n$.

Show that h_n converges to a unit vector h, and that $\phi(h) = 1$.

Show that h is orthogonal to N, the null space of ϕ, and also that $H = N \oplus \operatorname{span}(h)$.

Show that $\phi(k) = \langle k, h\rangle$, for all $k \in H$.

3H Groups, Rings and Fields

(i) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(ii) What are the units in $\mathbb{Z}[i]$? What are the primes in $\mathbb{Z}[i]$? Justify your answers.

Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

4F Dynamics of Differential Equations

(i) Define the terms *stable manifold* and *unstable manifold* of a hyperbolic fixed point $\mathbf{x}_0$ of a dynamical system. State carefully the stable manifold theorem.

Give an approximation, correct to fourth order in $|\mathbf{x}|$, for the stable and unstable manifolds of the origin for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x + x^2 - y^2 \\ -y + x^2 \end{pmatrix} .$$

(ii) State, without proof, the centre manifold theorem. Show that the fixed point at the origin of the system

$$\begin{aligned} \dot{x} &= y - x + ax^3 , \\ \dot{y} &= rx - y - zy , \\ \dot{z} &= -z + xy , \end{aligned}$$

where a is a constant, is non-hyperbolic at $r = 1$.

Using new coordinates $v = x + y$, $w = x - y$, find the centre manifold in the form

$$w = \alpha v^3 + \dots , \quad z = \beta v^2 + \gamma v^4 + \dots$$

for constants α, β, γ to be determined. Hence find the evolution equation on the centre manifold in the form

$$\dot{v} = \frac{1}{8}(a-1)v^3 + \left(\frac{(3a+1)(a+1)}{128} + \frac{(a-1)}{32} \right) v^5 + \dots .$$

Ignoring higher order terms, give conditions on a that guarantee that the origin is asymptotically stable.

5H Combinatorics

State and prove the local LYM inequality. Explain carefully when equality holds.

Define the colex order and state the Kruskal-Katona theorem. Deduce that, if n and r are fixed positive integers with $1 \leqslant r \leqslant n-1$, then for every $1 \leqslant m \leqslant \binom{n}{r}$ we have

$$\min\{|\partial\mathcal{A}| : \mathcal{A} \subset [n]^{(r)}, |\mathcal{A}| = m\} = \min\{|\partial\mathcal{A}| : \mathcal{A} \subset [n+1]^{(r)}, |\mathcal{A}| = m\}.$$

By a suitable choice of n, r and m, show that this result does not remain true if we replace the lower shadow ∂A with the upper shadow $\partial^+\mathcal{A}$.

6J Representation Theory

State and prove Schur's Lemma. Deduce that the centre of a finite group G with a faithful irreducible complex representation ρ is cyclic and that $Z(\rho(G))$ consists of scalar transformations.

Let G be the subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of dimension 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible complex representations.

Show finally that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

7K Differentiable Manifolds

State, giving your reasons, whether the following are true or false.

(a) Diffeomorphic connected manifolds must have the same dimension.

(b) Every non-zero vector bundle has a nowhere-zero section.

(c) Every projective space admits a volume form.

(d) If a manifold M has Euler characteristic zero, then M is orientable.

8J Algebraic Topology

Show that the fundamental group G of the Klein bottle is infinite. Show that G contains an abelian subgroup of finite index. Show that G is not abelian.

9J Number Fields

Let $K = \mathbb{Q}(\sqrt{35})$. By Dedekind's theorem, or otherwise, show that the ideal equations

$$2 = [2, \omega]^2, \quad 5 = [5, \omega]^2, \quad [\omega] = [2, \omega][5, \omega]$$

hold in K, where $\omega = 5 + \sqrt{35}$. Deduce that K has class number 2.

Verify that $1 + \omega$ is the fundamental unit in K. Hence show that the complete solution in integers x, y of the equation $x^2 - 35y^2 = -10$ is given by

$$x + \sqrt{35}y = \pm\omega(1 + \omega)^n \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Calculate the particular solution x, y for $n = 1$.
[*It can be assumed that the Minkowski constant for K is $\frac{1}{2}$.*]

10K Algebraic Curves

For $N \geq 1$, let V_N be the (irreducible) projective plane curve $V_N : X^N + Y^N + Z^N = 0$ over an algebraically closed field of characteristic zero.

Show that V_N is smooth (non-singular). For $m, n \geq 1$, let $\alpha_{m,n} : V_{mn} \to V_m$ be the morphism $\alpha_{m,n}(X : Y : Z) = (X^n : Y^n : Z^n)$. Determine the degree of $\alpha_{m,n}$, its points of ramification and the corresponding ramification indices.

Applying the Riemann–Hurwitz formula to $\alpha_{1,n}$, determine the genus of V_n.

11J Logic, Computation and Set Theory

Explain what is meant by a *structure* for a first-order language and by a *model* for a first-order theory. If T is a first-order theory whose axioms are all universal sentences (that is, sentences of the form $(\forall x_1 \ldots x_n)p$ where p is quantifier-free), show that every substructure of a T-model is a T-model.

Now let T be an arbitrary first-order theory in a language L, and let M be an L-structure satisfying all the universal sentences which are derivable from the axioms of T. If p is a quantifier-free formula (with free variables $x_1, \ldots, x_n$ say) whose interpretation $[p]_M$ is a nonempty subset of M^n, show that $T \cup \{(\exists x_1 \cdots x_n)p\}$ is consistent.

Let L' be the language obtained from L by adjoining a new constant $\widehat{a}$ for each element a of M, and let

$$T' = T \cup \{p[\widehat{a}_1, \ldots, \widehat{a}_n / x_1, \ldots, x_n] \,|\, p \text{ is quantifier-free and } (a_1, \ldots, a_n) \in [p]_M\}.$$

Show that T' has a model. [*You may use the Completeness and Compactness Theorems.*] Explain briefly why any such model contains a substructure isomorphic to M.

12L Probability and Measure

Let (X_n) be a sequence of non-negative random variables on a common probability space with $\mathbb{E}X_n \leqslant 1$, such that $X_n \to 0$ almost surely. Determine which of the following statements are necessarily true, justifying your answers carefully:

(a) $\mathbb{P}(X_n \geqslant 1) \to 0$ as $n \to \infty$;

(b) $\mathbb{E}X_n \to 0$ as $n \to \infty$;

(c) $\mathbb{E}(\sin(X_n)) \to 0$ as $n \to \infty$;

(d) $\mathbb{E}(\sqrt{X_n}) \to 0$ as $n \to \infty$.

[*Standard limit theorems for integrals, and results about uniform integrability, may be used without proof provided that they are clearly stated.*]

13L Applied Probability

Two enthusiastic probability students, Ros and Guil, sit an examination which starts at time 0 and ends at time T; they both decide to use the time to attempt a proof of a difficult theorem which carries a lot of extra marks.

Ros' strategy is to write the proof continuously at a constant speed λ lines per unit time. In a time interval of length δt he has a probability $\mu\delta t + o(\delta t)$ of realising he has made a mistake. If that happens he instantly panics, erases everything he has written and starts all over again.

Guil, on the other hand, keeps cool and thinks carefully about what he is doing. In a time interval of length δt, he has a probability $\lambda\delta t + o(\delta t)$ of writing the next line of proof and for each line he has written a probability $\mu\delta t + o(\delta t)$ of finding a mistake in that line, independently of all other lines he has written. When a mistake is found, he erases that line and carries on as usual, hoping for the best.

Both Ros and Guil realise that, even if they manage to finish the proof, they will not recognise that they have done so and will carry on writing as much as they can.

(a) Calculate $p_l(t)$, the probability that, for Ros, the length of his completed proof at time $t \geqslant l/\lambda$ is at least l.

(b) Let $q_n(t)$ be the probability that Guil has n lines of proof at time $t > 0$. Show that

$$\frac{\partial Q}{\partial t} = (s-1)(\lambda Q - \mu \frac{\partial Q}{\partial s}),$$

where $Q(s,t) = \sum_{n=0}^{\infty} s^n q_n(t)$.

(c) Suppose now that every time Ros starts all over again, the time until the next mistake has distribution F, independently of the past history. Write down a renewal-type integral equation satisfied by $l(t)$, the expected length of Ros' proof at time t. What is the expected length of proof produced by him at the end of the examination if F is the exponential distribution with mean $1/\mu$?

(d) What is the expected length of proof produced by Guil at the end of the examination if each line that he writes survives for a length of time with distribution F, independently of all other lines?

14M Information Theory

Define the binary Hamming code of length $n = 2^\ell - 1$ and its dual. Prove that the Hamming code is perfect. Prove that in the dual code:

(i) The weight of any non-zero codeword equals $2^{\ell-1}$;

(ii) The distance between any pair of words equals $2^{\ell-1}$.

[*You may quote results from the course provided that they are carefully stated.*]

15L Optimization and Control

State Pontryagin's maximum principle (PMP) for the problem of minimizing

$$\int_0^T c(x(t), u(t))\, dt + K(x(T))\,,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $dx/dt = a(x(t), u(t))$; here, $x(0)$ and T are given, and $x(T)$ is unconstrained.

Consider the two-dimensional problem in which $dx_1/dt = x_2$, $dx_2/dt = u$, $c(x,u) = \frac{1}{2}u^2$ and $K(x(T)) = \frac{1}{2}qx_1(T)^2$, $q > 0$. Show that, by use of a variable $z(t) = x_1(t) + x_2(t)(T-t)$, one can rewrite this problem as an equivalent one-dimensional problem.

Use PMP to solve this one-dimensional problem, showing that the optimal control can be expressed as $u(t) = -qz(T)(T-t)$, where $z(T) = z(0)/(1 + \frac{1}{3}qT^3)$.

Express $u(t)$ in a feedback form of $u(t) = k(t)z(t)$ for some $k(t)$.

Suppose that the initial state $x(0)$ is perturbed by a small amount to $x(0) + (\epsilon_1, \epsilon_2)$. Give an expression (in terms of ϵ_1, ϵ_2, $x(0)$, q and T) for the increase in minimal cost.

16M Principles of Statistics

(i) Let X be a random variable with density function $f(x;\theta)$. Consider testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1$.

What is the form of the optimal size α classical hypothesis test?

Compare the form of the test with the Bayesian test based on the Bayes factor, *and* with the Bayes decision rule under the 0-1 loss function, under which a loss of 1 is incurred for an incorrect decision and a loss of 0 is incurred for a correct decision.

(ii) What does it mean to say that a family of densities $\{f(x;\theta), \theta \in \Theta\}$ with real scalar parameter θ is of *monotone likelihood ratio*?

Suppose X has a distribution from a family which is of monotone likelihood ratio with respect to a statistic $t(X)$ and that it is required to test $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$.

State, without proof, a theorem which establishes the existence of a uniformly most powerful test and describe in detail the form of the test.

Let $X_1, \ldots, X_n$ be independent, identically distributed $U(0,\theta)$, $\theta > 0$. Find a uniformly most powerful size α test of $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$, and find its power function. Show that we may construct a different, randomised, size α test with the same power function for $\theta \geqslant \theta_0$.

17G Partial Differential Equations

(a) Define the convolution $f * g$ of two functions. Write down a formula for a solution $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ to the initial value problem

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

together with the boundary condition

$$u(0, x) = f(x)$$

for f a bounded continuous function on $\mathbb{R}^n$. Comment briefly on the uniqueness of the solution.

(b) State and prove the Duhamel principle giving the solution (for $t > 0$) to the equation

$$\frac{\partial u}{\partial t} - \Delta u = g$$

together with the boundary condition

$$u(0, x) = f(x)$$

in terms of your answer to (a).

(c) Show that if $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is the solution to

$$\frac{\partial v}{\partial t} - \Delta v = G$$

together with the boundary condition

$$v(0, x) = f(x)$$

with $G(t, x) \leq g(t, x)$ for all (t, x) then $v(t, x) \leq u(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$.

Finally show that if in addition there exists a point (t_0, x_0) at which there is strict inequality in the assumption i.e.

$$G(t_0, x_0) < g(t_0, x_0),$$

then in fact

$$v(t, x) < u(t, x)$$

whenever $t > t_0$.

18G Methods of Mathematical Physics

Show that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{t^{z-1}}{t-a}\, dt = \pi i a^{z-1} ,$$

where a is real and positive, $0 < \mathrm{Re}\,(z) < 1$ and $\mathcal{P}$ denotes the Cauchy principal value; the principal branches of t^z etc. are implied. Deduce that

$$\int_0^{\infty} \frac{t^{z-1}}{t+a}\, dt = \pi a^{z-1} \mathrm{cosec}\, \pi z \qquad (*)$$

and that

$$\mathcal{P}\int_0^{\infty} \frac{t^{z-1}}{t-a}\, dt = -\pi a^{z-1} \cot \pi z \ .$$

Use $(*)$ to show that, if $\mathrm{Im}\,(b) > 0$, then

$$\int_0^{\infty} \frac{t^{z-1}}{t-b}\, dt = -\pi b^{z-1} (\cot \pi z - i) \ .$$

What is the value of this integral if $\mathrm{Im}\,(b) < 0$?

19F Numerical Analysis

(i)

Given the finite-difference method

$$\sum_{k=-r}^{s} \alpha_k u_{m+k}^{n+1} = \sum_{k=-r}^{s} \beta_k u_{m+k}^{n}, \quad m, n \in \mathbb{Z},\ n \geqslant 0,$$

define

$$H(z) = \frac{\sum\limits_{k=-r}^{s} \beta_k z^k}{\sum\limits_{k=-r}^{s} \alpha_k z^k}.$$

Prove that this method is stable if and only if

$$|H(e^{i\theta})| \leqslant 1, \quad -\pi \leqslant \theta \leqslant \pi.$$

[*You may quote without proof known properties of the Fourier transform.*]

(ii) Find the range of the parameter μ such that the method

$$(1-2\mu) u_{m-1}^{n+1} + 4\mu u_m^{n+1} + (1-2\mu) u_{m+1}^{n+1} = u_{m-1}^{n} + u_{m+1}^{n}$$

is stable. Supposing that this method is used to solve the diffusion equation for $u(x,t)$, determine the order of magnitude of the local error as a power of Δx.

20E Electrodynamics

A particle of rest mass m and charge q moves in an electromagnetic field given by a potential A_a along a trajectory $x^a(\tau)$, where τ is the proper time along the particle's worldline. The action for such a particle is

$$I = \int \left(m\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b} - qA_a\dot{x}^a \right) d\tau \, .$$

Show that the Euler-Lagrange equations resulting from this action reproduce the relativistic equation of motion for the particle.

Suppose that the particle is moving in the electrostatic field of a fixed point charge Q with radial electric field E_r given by

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} \, .$$

Show that one can choose a gauge such that $A_i = 0$ and only $A_0 \neq 0$. Find A_0.

Assume that the particle executes planar motion, which in spherical polar coordinates $(r\,,\theta\,,\phi)$ can be taken to be in the plane $\theta = \pi/2$. Derive the equations of motion for t and ϕ.

By using the fact that $\eta_{ab}\dot{x}^a\dot{x}^b = -1$, find the equation of motion for r, and hence show that the shape of the orbit is described by

$$\frac{dr}{d\phi} = \pm\frac{r^2}{\ell}\sqrt{\left(E + \frac{\gamma}{r}\right)^2 - 1 - \frac{\ell^2}{r^2}} \, ,$$

where E (> 1) and ℓ are constants of integration and γ is to be determined.

By putting $u = 1/r$ or otherwise, show that if $\gamma^2 < \ell^2$ then the orbits are bounded and generally not closed, and show that the angle between successive minimal values of r is $2\pi(1 - \gamma^2/\ell^2)^{-1/2}$.

21E Foundations of Quantum Mechanics

(i) A Hamiltonian H_0 has energy eigenvalues E_r and corresponding non-degenerate eigenstates $|r\rangle$. Show that under a small change in the Hamiltonian δH,

$$\delta|r\rangle = \sum_{s \neq r} \frac{\langle s|\delta H|r\rangle}{E_r - E_s} |s\rangle,$$

and derive the related formula for the change in the energy eigenvalue E_r to first and second order in δH.

(ii) The Hamiltonian for a particle moving in one dimension is $H = H_0 + \lambda H'$, where $H_0 = p^2/2m + V(x)$, $H' = p/m$ and λ is small. Show that

$$\frac{i}{\hbar}[H_0, x] = H'$$

and hence that

$$\delta E_r = -\lambda^2 \frac{i}{\hbar} \langle r|H'x|r\rangle = \lambda^2 \frac{i}{\hbar} \langle r|xH'|r\rangle$$

to second order in λ.

Deduce that δE_r is independent of the particular state $|r\rangle$ and explain why this change in energy is exact to all orders in λ.

22E Applications of Quantum Mechanics

Define the reciprocal lattice for a lattice L with lattice vectors $\boldsymbol{\ell}$.

A beam of electrons, with wave vector $\mathbf{k}$, is incident on a Bravais lattice L with a large number of atoms, N. If the scattering amplitude for scattering on an individual atom in the direction $\hat{\mathbf{k}}'$ is $f(\hat{\mathbf{k}}')$, show that the scattering amplitude for the whole lattice is

$$\sum_{\boldsymbol{\ell} \in L} e^{i\mathbf{q}\cdot\boldsymbol{\ell}} f(\hat{\mathbf{k}}') , \qquad \mathbf{q} = \mathbf{k} - |\mathbf{k}|\, \hat{\mathbf{k}}' .$$

Derive the formula for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = N |f(\hat{\mathbf{k}}')|^2 \Delta(\mathbf{q}) ,$$

obtaining an explicit form for $\Delta(\mathbf{q})$. Show that $\Delta(\mathbf{q})$ is strongly peaked when $\mathbf{q} = \mathbf{g}$, a reciprocal lattice vector. Show that this leads to the Bragg formula $2d \sin \frac{\theta}{2} = \lambda$, where θ is the scattering angle, λ the electron wavelength and d the separation between planes of atoms in the lattice.

23D General Relativity

(i) Consider the line element describing the interior of a star,

$$ds^2 = e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{2\gamma(r)}dt^2 \ ,$$

defined for $0 \le r \le r_0$ by

$$e^{-2\alpha(r)} = 1 - Ar^2$$

and

$$e^{\gamma(r)} = \frac{3}{2}e^{-\alpha_0} - \frac{1}{2}e^{-\alpha(r)}.$$

Here $A = 2M/r_0^3$, M is the mass of the star, and α_0 is defined to be $\alpha(r_0)$.

The star is made of a perfect fluid with energy-momentum tensor

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab}.$$

Here u^a is the 4-velocity of the fluid which is at rest, the density ρ is constant throughout the star ($0 \le r \le r_0$) and the pressure $p = p(r)$ depends only on the radial coordinate. Write down the Einstein field equations and show that (in geometrical units with $G = c = 1$) they may equivalently be written as

$$R_{ab} = 8\pi(p+\rho)u_a u_b + 4\pi(p-\rho)g_{ab}.$$

(ii) Using the formulae below, or otherwise, show that for $0 \le r \le r_0$ one has

$$\rho = \frac{3A}{8\pi}, \quad p(r) = \frac{3A}{8\pi}\left(\frac{e^{-\alpha(r)} - e^{-\alpha_0}}{3e^{-\alpha_0} - e^{-\alpha(r)}}\right).$$

[*The non-zero components of the Ricci tensor are:*

$$R_{11} = -\gamma'' + \alpha'\gamma' - \gamma'^2 + \frac{2\alpha'}{r}, \quad R_{22} = e^{-2\alpha}[(\alpha'-\gamma')r - 1] + 1,$$

$$R_{33} = \sin^2\theta R_{22}, \quad R_{44} = e^{2\gamma-2\alpha}[\gamma'' - \alpha'\gamma' + \gamma'^2 + \frac{2\gamma'}{r}].$$

Note that

$$\alpha' = Are^{2\alpha}, \quad \gamma' = \frac{1}{2}Are^{\alpha-\gamma}, \quad \gamma'' = \frac{1}{2}Ae^{\alpha-\gamma} + \frac{1}{2}A^2r^2e^{3\alpha-\gamma} - \frac{1}{4}A^2r^2e^{2\alpha-2\gamma}. \quad]$$

24C Fluid Dynamics II

A thin layer of liquid of kinematic viscosity ν flows under the influence of gravity down a plane inclined at an angle α to the horizontal $(0 \leq \alpha \leq \pi/2)$. With origin O on the plane, and axes Ox down the line of steepest slope and Oy normal to the plane, the free surface is given by $y = h(x,t)$, where $|\partial h/\partial x| \ll 1$. The pressure distribution in the liquid may be assumed to be hydrostatic. Using the approximations of lubrication theory, show that

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left\{h^3(\cos\alpha\frac{\partial h}{\partial x} - \sin\alpha)\right\}.$$

Now suppose that

$$h = h_0 + \eta(x,t) \ ,$$

where

$$\eta(x,0) = \eta_0 e^{-x^2/a^2}$$

and h_0, η_0 and a are constants with $\eta_0 \ll a, h_0$. Show that, to leading order,

$$\eta(x,t) = \frac{a\eta_0}{(a^2+4Dt)^{1/2}}\exp\left\{-\frac{(x-Ut)^2}{a^2+4Dt}\right\} \ ,$$

where U and D are constants to be determined.

Explain in physical terms the meaning of this solution.

25C Waves in Fluid and Solid Media

Starting from the equations for one-dimensional unsteady flow of a perfect gas of uniform entropy, show that the Riemann invariants,

$$R_\pm = u \pm \frac{2}{\gamma-1}(c-c_0) \ ,$$

are constant on characteristics $C_\pm$ given by $\dfrac{dx}{dt} = u \pm c$, where $u(x,t)$ is the velocity of the gas, $c(x,t)$ is the local speed of sound and γ is the specific heat ratio.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas and the piston are initially at rest. At time $t = 0$ the piston starts moving to the left at a constant speed V. Find $u(x,t)$ and $c(x,t)$ in the three regions

(i) $c_0 t \leq x$,
(ii) $at \leq x < c_0 t$,
(iii) $-Vt \leq x < at$,

where $a = c_0 - \frac{1}{2}(\gamma+1)V$. What is the largest value of V for which c is positive throughout region (iii)? What happens if V exceeds this value?