MATHEMATICAL TRIPOS Part II Alternative A

Thursday 6 June 2002 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than ***FOUR*** *questions. If you submit answers to more than four questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***19C, 21C*** *should be in one bundle and* ***12L, 14L*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

Write an essay on the long-time behaviour of discrete-time Markov chains on a finite state space. Your essay should include discussion of the convergence of probabilities as well as almost-sure behaviour. You should also explain what happens when the chain is not irreducible.

2F Principles of Dynamics

Explain how the orientation of a rigid body can be specified by means of the three Eulerian angles, θ, ϕ and ψ.

An axisymmetric top of mass M has principal moments of inertia A, A and C, and is spinning with angular speed n about its axis of symmetry. Its centre of mass lies a distance h from the fixed point of support. Initially the axis of symmetry points vertically upwards. It then suffers a small disturbance. For what values of the spin is the initial configuration stable?

If the spin is such that the initial configuration is unstable, what is the lowest angle reached by the symmetry axis in the nutation of the top? Find the maximum and minimum values of the precessional angular velocity $\dot{\phi}$.

3K Functional Analysis

Define the *distribution function* Φ_f of a non-negative measurable function f on the interval $I = [0,1]$. Show that Φ_f is a decreasing non-negative function on $[0,\infty]$ which is continuous on the right.

Define the *Lebesgue integral* $\int_I f\,dm$. Show that $\int_I f\,dm = 0$ if and only if $f = 0$ almost everywhere.

Suppose that f is a non-negative Riemann integrable function on $[0,1]$. Show that there are an increasing sequence (g_n) and a decreasing sequence (h_n) of non-negative step functions with $g_n \leqslant f \leqslant h_n$ such that $\int_0^1 (h_n(x) - g_n(x))\,dx \to 0$.

Show that the functions $g = \lim_n g_n$ and $h = \lim_n h_n$ are equal almost everywhere, that f is measurable and that the Lebesgue integral $\int_I f\,dm$ is equal to the Riemann integral $\int_0^1 f(x)\,dx$.

Suppose that j is a Riemann integrable function on $[0,1]$ and that $j(x) > 0$ for all x. Show that $\int_0^1 j(x)\,dx > 0$.

4H Groups, Rings and Fields

Let F be a finite field. Show that there is a unique prime p for which F contains the field $\mathbb{F}_p$ of p elements. Prove that F contains p^n elements, for some $n \in \mathbb{N}$. Show that $x^{p^n} = x$ for all $x \in F$, and hence find a polynomial $f \in \mathbb{F}_p[X]$ such that F is the splitting field of f. Show that, up to isomorphism, F is the unique field $\mathbb{F}_{p^n}$ of size p^n.

[*Standard results about splitting fields may be assumed.*]

Prove that the mapping sending x to x^p is an automorphism of $\mathbb{F}_{p^n}$. Deduce that the Galois group Gal $(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic of order n. For which m is $\mathbb{F}_{p^m}$ a subfield of $\mathbb{F}_{p^n}$?

5D Electromagnetism

State the four integral relationships between the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ and explain their physical significance. Derive Maxwell's equations from these relationships and show that $\mathbf{E}$ and $\mathbf{B}$ can be described by a scalar potential ϕ and a vector potential $\mathbf{A}$ which satisfy the inhomogeneous wave equations

$$\nabla^2 \phi - \epsilon_0 \mu_0 \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0},$$

$$\nabla^2 \mathbf{A} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}.$$

If the current $\mathbf{j}$ satisfies Ohm's law and the charge density $\rho = 0$, show that plane waves of the form

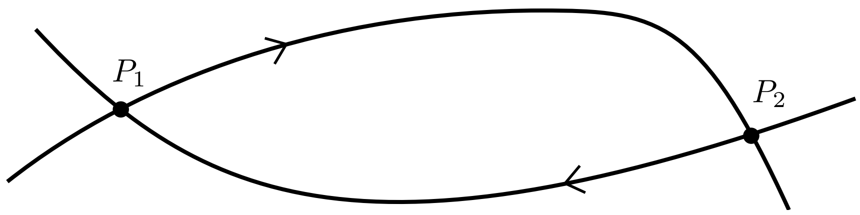
$$\mathbf{A} = A(z,t) e^{i\omega t} \hat{\mathbf{x}},$$

where $\hat{\mathbf{x}}$ is a unit vector in the x-direction of cartesian axes (x, y, z), are damped. Find an approximate expression for $A(z,t)$ when $\omega \ll \sigma/\epsilon_0$, where σ is the electrical conductivity.

6F Dynamics of Differential Equations

Define the terms *homoclinic orbit*, *heteroclinic orbit* and *heteroclinic loop*. In the case of a dynamical system that possesses a homoclinic orbit, explain, without detailed calculation, how to calculate its stability.

A second order dynamical system depends on two parameters μ_1 and μ_2. When $\mu_1 = \mu_2 = 0$ there is a heteroclinic loop between the points P_1, P_2 as in the diagram.



When μ_1, μ_2 are small there are trajectories that pass close to the fixed points P_1, P_2:

By adapting the method used above for trajectories near homoclinic orbits, show that the distances y_n, y_{n+1} to the stable manifold at P_1 on successive returns are related to z_n, z_{n+1}, the corresponding distances near P_2, by coupled equations of the form

$$\left.\begin{aligned} z_n &= (y_n)^{\gamma_1} + \mu_1, \\ y_{n+1} &= (z_n)^{\gamma_2} + \mu_2, \end{aligned}\right\}$$

where any arbitrary constants have been removed by rescaling, and γ_1, γ_2 depend on conditions near P_1, P_2. Show from these equations that there is a stable heteroclinic orbit ($\mu_1 = \mu_2 = 0$) if $\gamma_1\gamma_2 > 1$. Show also that in the marginal situation $\gamma_1 = 2$, $\gamma_2 = \frac{1}{2}$ there can be a stable fixed point for small positive y, z if $\mu_2 < 0$, $\mu_2^2 < \mu_1$. Explain carefully the form of the orbit of the original dynamical system represented by the solution of the above map when $\mu_2^2 = \mu_1$.

7K Geometry of Surfaces

Write an essay on the Euler number of topological surfaces. Your essay should include a definition of subdivision, some examples of surfaces and their Euler numbers, and a discussion of the statement and significance of the Gauss–Bonnet theorem.

8J Logic, Computation and Set Theory

Let P be a set of primitive propositions. Let $L(P)$ denote the set of all compound propositions over P, and let S be a subset of $L(P)$. Consider the relation $\preceq_S$ on $L(P)$ defined by

$$s \preceq_S t \text{ if and only if } S \cup \{s\} \vdash t.$$

Prove that $\preceq_S$ is reflexive and transitive. Deduce that if we define $\approx_S$ by ($s \approx_S t$ if and only if $s \preceq_S t$ and $t \preceq_S s$), then $\approx_S$ is an equivalence relation and the quotient $B_S = L(P)/\approx_S$ is partially ordered by the relation $\leqslant_S$ induced by $\preccurlyeq_S$ (that is, $[s] \leqslant_S [t]$ if and only if $s \preccurlyeq_S t$, where square brackets denote equivalence classes).

Assuming the result that B_S is a Boolean algebra with lattice operations induced by the logical operations on $L(P)$ (that is, $[s] \wedge [t] = [s \wedge t]$, etc.), show that there is a bijection between the following two sets:
(a) The set of lattice homomorphisms $B_S \to \{0, 1\}$.
(b) The set of models of the propositional theory S.

Deduce that the completeness theorem for propositional logic is equivalent to the assertion that, for any Boolean algebra B with more than one element, there exists a homomorphism $B \to \{0, 1\}$.

[*You may assume the result that the completeness theorem implies the compactness theorem.*]

9H Graph Theory

Write an essay on connectivity in graphs.

Your essay should include proofs of at least two major theorems, along with a discussion of one or two significant corollaries.

10J Number Theory

Write an essay on quadratic reciprocity. Your essay should include (i) a proof of the law of quadratic reciprocity for the Legendre symbol, (ii) a proof of the law of quadratic reciprocity for the Jacobi symbol, and (iii) a comment on why this latter law is useful in primality testing.

11M Algorithms and Networks

Write an essay on Strong Lagrangian problems. You should give an account of duality and how it relates to the Strong Lagrangian property. In particular, establish carefully the relationship between the Strong Lagrangian property and supporting hyperplanes.

Also, give an example of a class of problems that are Strong Lagrangian. [*You should explain carefully why your example has the Strong Lagrangian property.*]

12L Stochastic Financial Models

Write an essay on the Black–Scholes formula for the price of a European call option on a stock. Your account should include a derivation of the formula and a careful analysis of its dependence on the parameters of the model.

13M Principles of Statistics

(a) Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from a one-parameter distribution with density function

$$f(x;\theta) = h(x)g(\theta)\exp\{\theta t(x)\}\ ,\ x \in \mathbb{R}.$$

Explain in detail how you would test

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta \neq \theta_0\ .$$

What is the general form of a conjugate prior density for θ in a Bayesian analysis of this distribution?

(b) Let Y_1, Y_2 be independent Poisson random variables, with means $(1-\psi)\lambda$ and $\psi\lambda$ respectively, with λ *known*.

Explain why the Conditionality Principle leads to inference about ψ being drawn from the conditional distribution of Y_2, given Y_1+Y_2. What is this conditional distribution?

(c) Suppose Y_1, Y_2 have distributions as in (b), but that λ is now *unknown*.

Explain in detail how you would test $H_0 : \psi = \psi_0$ against $H_1 : \psi \neq \psi_0$, and describe the optimality properties of your test.

[*Any general results you use should be stated clearly, but need not be proved.*]

14L Computational Statistics and Statistical Modelling

Assume that the n-dimensional observation vector Y may be written as $Y = X\beta + \epsilon$, where X is a given $n \times p$ matrix of rank p, β is an unknown vector, with $\beta^T = (\beta_1, \ldots, \beta_p)$, and

$$\epsilon \sim N_n(0, \sigma^2 I) \qquad (*)$$

where σ^2 is unknown. Find $\hat{\beta}$, the least-squares estimator of β, and describe (without proof) how you would test

$$H_0 : \beta_\nu = 0$$

for a given ν.

Indicate briefly two plots that you could use as a check of the assumption $(*)$.

Continued opposite

Sulphur dioxide is one of the major air pollutants. A data-set presented by Sokal and Rohlf (1981) was collected on 41 US cities in 1969-71, corresponding to the following variables:

Y = sulphur dioxide content of air in micrograms per cubic metre

$X1$ = average annual temperature in degrees Fahrenheit

$X2$ = number of manufacturing enterprises employing 20 or more workers

$X3$ = population size (1970 census) in thousands

$X4$ = average annual wind speed in miles per hour

$X5$ = average annual precipitation in inches

$X6$ = average annual of days with precipitation per year.

Interpret the R output that follows below, quoting any standard theorems that you need to use.

> next.lm _ lm(log(Y) ~ X1 + X2 + X3 + X4 + X5 + X6)

> summary(next.lm)

Call: lm(formula = log(Y) ~ X1 + X2 + X3 + X4 + X5 + X6)

```
Residuals:
      Min        1Q    Median        3Q       Max
 -0.79548  -0.25538  -0.01968   0.28328   0.98029
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> |t|)	
(Intercept)	7.2532456	1.4483686	5.008	1.68e-05	***
X1	-0.0599017	0.0190138	-3.150	0.00339	**
X2	0.0012639	0.0004820	2.622	0.01298	*
X3	-0.0007077	0.0004632	-1.528	0.13580	
X4	-0.1697171	0.0555563	-3.055	0.00436	**
X5	0.0173723	0.0111036	1.565	0.12695	
X6	0.0004347	0.0049591	0.088	0.93066	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'

Residual standard error: 0.448 on 34 degrees of freedom

Multiple R-Squared: 0.6541

F-statistic: 10.72 on 6 and 34 degrees of freedom, p-value: 1.126e-06

15E Foundations of Quantum Mechanics

Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have (a) spin zero, (b) spin 1/2.

The stationary Schrödinger equation for one particle in the potential

$$-\frac{2e^2}{4\pi\epsilon_0 r}$$

has normalized, spherically symmetric, real wave functions $\psi_n(\mathbf{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. What are the consequences of the Pauli exclusion principle for the ground state of the helium atom? Assuming that wavefunctions which are not spherically symmetric can be ignored, what are the states of the first excited energy level of the helium atom?
[*You may assume here that the electrons are non-interacting.*]

Show that, taking into account the interaction between the two electrons, the estimate for the energy of the ground state of the helium atom is

$$2E_0 + \frac{e^2}{4\pi\epsilon_0} \int \frac{d^3\mathbf{r}_1\, d^3\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \psi_0^2(\mathbf{r}_1)\psi_0^2(\mathbf{r}_2).$$

16E Quantum Physics

Explain how the energy band structure for electrons determines the conductivity properties of crystalline materials.

A semiconductor has a conduction band with a lower edge E_c and a valence band with an upper edge E_v . Assuming that the density of states for electrons in the conduction band is

$$\rho_c(E) = B_c(E - E_c)^{\frac{1}{2}} \ , \qquad E > E_c \ \ ,$$

and in the valence band is

$$\rho_v(E) = B_v(E_v - E)^{\frac{1}{2}} \ , \qquad E < E_v \ \ ,$$

where B_c and B_v are constants characteristic of the semiconductor, explain why at low temperatures the chemical potential for electrons lies close to the mid-point of the gap between the two bands.

Describe what is meant by the doping of a semiconductor and explain the distinction between n-type and p-type semiconductors, and discuss the low temperature limit of the chemical potential in both cases. Show that, whatever the degree and type of doping,

$$n_e n_p = B_c B_v [\Gamma(3/2)]^2 (kT)^3 e^{-(E_c - E_v)/kT} \ \ ,$$

where n_e is the density of electrons in the conduction band and n_p is the density of holes in the valence band.

17D General Relativity

With respect to the Schwarzschild coordinates (r, θ, ϕ, t), the Schwarzschild geometry is given by

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - \left(1 - \frac{r_s}{r}\right) dt^2,$$

where $r_s = 2M$ is the Schwarzschild radius and M is the Schwarzschild mass. Show that, by a suitable choice of (θ, ϕ), the general geodesic can regarded as moving in the equatorial plane $\theta = \pi/2$. Obtain the equations governing timelike and null geodesics in terms of $u(\phi)$, where $u = 1/r$.

Discuss light bending and perihelion precession in the solar system.

18D Statistical Physics and Cosmology

What is an ideal gas? Explain how the microstates of an ideal gas of indistinguishable particles can be labelled by a set of integers. What range of values do these integers take for (a) a boson gas and (b) a Fermi gas?

Let E_i be the energy of the i-th one-particle energy eigenstate of an ideal gas in thermal equilibrium at temperature T and let $p_i(n_i)$ be the probability that there are n_i particles of the gas in this state. Given that

$$p_i(n_i) = e^{-\beta E_i n_i}/Z_i \quad (\beta = \frac{1}{kT}),$$

determine the normalization factor Z_i for (a) a boson gas and (b) a Fermi gas. Hence obtain an expression for $\bar{n}_i$, the average number of particles in the i-th one-particle energy eigenstate for both cases (a) and (b).

In the case of a Fermi gas, write down (without proof) the generalization of your formula for $\bar{n}_i$ to a gas at non-zero chemical potential μ. Show how it leads to the concept of a Fermi energy ϵ_F for a gas at zero temperature. How is ϵ_F related to the Fermi momentum p_F for (a) a non-relativistic gas and (b) an ultra-relativistic gas?

In an approximation in which the discrete set of energies E_i is replaced with a continuous set with momentum p, the density of one-particle states with momentum in the range p to $p + dp$ is $g(p)dp$. Explain briefly why

$$g(p) \propto p^2 V, \tag{$*$}$$

where V is the volume of the gas. Using this formula, obtain an expression for the total energy E of an ultra-relativistic gas at zero chemical potential as an integral over p. Hence show that

$$\frac{E}{V} \propto T^{\alpha},$$

where α is a number that you should compute. Why does this result apply to a photon gas?

Using the formula ($*$) for a non-relativistic Fermi gas at zero temperature, obtain an expression for the particle number density n in terms of the Fermi momentum and provide a physical interpretation of this formula in terms of the typical de Broglie wavelength. Obtain an analogous formula for the (internal) energy density and hence show that the pressure P behaves as

$$P \propto n^{\gamma}$$

where γ is a number that you should compute. [*You need not prove any relation between the pressure and the energy density you use.*] What is the origin of this pressure given that $T = 0$ by assumption? Explain briefly and qualitatively how it is relevant to the stability of white dwarf stars.

19C Transport Processes

(a) A biological vessel is modelled two-dimensionally as a fluid-filled channel bounded by parallel plane walls $y = \pm a$, embedded in an infinite region of fluid-saturated tissue. In the tissue a solute has concentration $C^{out}(y,t)$, diffuses with diffusivity D and is consumed by biological activity at a rate kC^{out} per unit volume, where D and k are constants. By considering the solute balance in a slice of tissue of infinitesimal thickness, show that

$$C^{out}_t = DC^{out}_{yy} - kC^{out}.$$

A *steady* concentration profile $C^{out}(y)$ results from a flux $\beta\left(C^{in} - C^{out}_a\right)$, per unit area of wall, of solute from the channel into the tissue, where C^{in} is a constant concentration of solute that is maintained in the channel and $C^{out}_a = C^{out}(a)$. Write down the boundary conditions satisfied by $C^{out}(y)$. Solve for $C^{out}(y)$ and show that

$$C^{out}_a = \frac{\gamma}{\gamma+1}C^{in}, \qquad (*)$$

where $\gamma = \beta/\sqrt{kD}$.

(b) Now let the solute be supplied by steady flow down the channel from one end, $x = 0$, with the channel taken to be semi-infinite in the x-direction. The cross-sectionally averaged velocity in the channel $u(x)$ varies due to a flux of fluid from the tissue to the channel (by osmosis) equal to $\lambda\left(C^{in} - C^{out}_a\right)$ per unit area. Neglect both the variation of $C^{in}(x)$ across the channel and diffusion in the x-direction.

By considering conservation of fluid, show that

$$au_x = \lambda\left(C^{in} - C^{out}_a\right)$$

and write down the corresponding equation derived from conservation of solute. Deduce that

$$u(\lambda C^{in} + \beta) = u_0(\lambda C^{in}_0 + \beta)\ ,$$

where $u_0 = u(0)$ and $C^{in}_0 = C^{in}(0)$.

Assuming that equation $(*)$ still holds, even though C^{out} is now a function of x as well as y, show that $u(x)$ satisfies the ordinary differential equation

$$(\gamma+1)auu_x + \beta u = u_0\left(\lambda C^{in}_0 + \beta\right).$$

Find scales $\hat{x}$ and $\hat{u}$ such that the dimensionless variables $U = u/\hat{u}$ and $X = x/\hat{x}$ satisfy

$$UU_X + U = 1.$$

Derive the solution $(1-U)e^U = Ae^{-X}$ and find the constant A.

To what values do u and C_{in} tend as $x \to \infty$?

20G Theoretical Geophysics

The equation of motion for small displacements $\mathbf{u}$ in a homogeneous, isotropic, elastic material is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \mu \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u}) \quad ,$$

where λ and μ are the Lamé constants. Derive the conditions satisfied by the polarisation $\mathbf{P}$ and (real) vector slowness $\mathbf{s}$ of plane-wave solutions $\mathbf{u} = \mathbf{P} f(\mathbf{s} \cdot \mathbf{x} - t)$, where f is an arbitrary scalar function. Describe the division of these waves into P-waves, SH-waves and SV-waves.

A plane harmonic SV-wave of the form

$$\mathbf{u} = (s_3, 0, -s_1) \exp[i\omega(s_1 x_1 + s_3 x_3 - t)]$$

travelling through homogeneous elastic material of P-wave speed α and S-wave speed β is incident from $x_3 < 0$ on the boundary $x_3 = 0$ of rigid material in $x_3 > 0$ in which the displacement is identically zero.

Write down the form of the reflected wavefield in $x_3 < 0$. Calculate the amplitudes of the reflected waves in terms of the components of the slowness vectors.

Derive expressions for the components of the incident and reflected slowness vectors, in terms of the wavespeeds and the angle of incidence θ_0. Hence show that there is no reflected SV-wave if

$$\sin^2 \theta_0 = \frac{\beta^2}{\alpha^2 + \beta^2} \quad .$$

Sketch the rays produced if the region $x_3 > 0$ is fluid instead of rigid.

21C Mathematical Methods

State Watson's lemma giving an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_1 = \int_0^A f(t) e^{-\lambda t} dt\,, \quad A > 0\ .$$

Show how this result may be used to find an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_2 = \int_{-A}^{B} f(t) e^{-\lambda t^2} dt\,, \quad A > 0,\, B > 0\ .$$

Hence derive Laplace's method for obtaining an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_3 = \int_a^b f(t) e^{\lambda \phi(t)} dt\ ,$$

where $\phi(t)$ is differentiable, for the cases: (i) $\phi'(t) < 0$ in $a \le t \le b$; and (ii) $\phi'(t)$ has a simple zero at $t = c$ with $a < c < b$ and $\phi''(c) < 0$.

Find the first two terms in the asymptotic expansion as $x \to \infty$ of

$$I_4 = \int_{-\infty}^{\infty} \log(1+t^2) e^{-xt^2} dt\ .$$

[*You may leave your answer expressed in terms of* Γ*-functions.*]

22F Numerical Analysis

Write an essay on the method of conjugate gradients. You should describe the algorithm, present an analysis of its properties and discuss its advantages.

[*Any theorems quoted should be stated precisely but need not be proved.*]