UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Wednesday 5 June 2002 9 to 12

PAPER 3

Before you begin read these instructions carefully.

Each question is divided into Part ***(i)*** *and Part* ***(ii)****, which may or may not be related. Candidates may attempt either or both Parts of any question, but must not attempt Parts from more than* ***SIX*** *questions. If you submit answers to Parts of more than six questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same, with Part ***(ii)*** *of each question carrying twice as many marks as Part* ***(i)****. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***13E, 15E*** *should be in one bundle and* ***6F, 18F*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *Parts of* ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) Consider the continuous-time Markov chain $(X_t)_{t\geqslant 0}$ on $\{1,2,3,4,5,6,7\}$ with generator matrix

$$Q = \begin{pmatrix} -6 & 2 & 0 & 0 & 0 & 4 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -6 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

Compute the probability, starting from state 3, that X_t hits state 2 eventually.

Deduce that

$$\lim_{t\to\infty} \mathbb{P}(X_t = 2 | X_0 = 3) = \frac{4}{15}.$$

[*Justification of standard arguments is not expected.*]

(ii) A colony of cells contains immature and mature cells. Each immature cell, after an exponential time of parameter 2, becomes a mature cell. Each mature cell, after an exponential time of parameter 3, divides into two immature cells. Suppose we begin with one immature cell and let $n(t)$ denote the expected number of immature cells at time t. Show that

$$n(t) = (4e^t + 3e^{-6t})/7.$$

2G Principles of Dynamics

(i) Show that Hamilton's equations follow from the variational principle

$$\delta \int_{t_1}^{t_2} [p\dot{q} - H(q,p,t)]\, dt = 0$$

under the restrictions $\delta q(t_1) = \delta q(t_2) = \delta p(t_1) = \delta p(t_2) = 0$. Comment on the difference from the variational principle for Lagrange's equations.

(ii) Suppose we transform from p and q to $p' = p'(q,p,t)$ and $q' = q'(q,p,t)$, with

$$p'\dot{q}' - H' = p\dot{q} - H + \frac{\mathrm{d}}{\mathrm{d}t} F(q,p,q',p',t),$$

where H' is the new Hamiltonian. Show that p' and q' obey Hamilton's equations with Hamiltonian H'.

Show that the time independent generating function $F = F_1(q,q') = q'/q$ takes the Hamiltonian

$$H = \frac{1}{2q^2} + \frac{1}{2}p^2 q^4$$

to harmonic oscillator form. Show that q' and p' obey the Poisson bracket relation

$$\{q', p'\} = 1.$$

3K Functional Analysis

(i) Suppose that (f_n) is a decreasing sequence of continuous real-valued functions on a compact metric space (X,d) which converges pointwise to 0. By considering sets of the form $B_n = \{x : f_n(x) < \epsilon\}$, for $\epsilon > 0$, or otherwise, show that f_n converges uniformly to 0.

Can the condition that (f_n) is decreasing be dropped? Can the condition that (X,d) is compact be dropped? Justify your answers.

(ii) Suppose that k is a positive integer. Define polynomials p_n recursively by

$$p_0 = 0, \qquad p_{n+1}(t) = p_n(t) + (t - p_n^k(t))/k.$$

Show that $0 \leqslant p_n(t) \leqslant p_{n+1}(t) \leqslant t^{1/k}$, for $t \in [0,1]$, and show that $p_n(t)$ converges to $t^{1/k}$ uniformly on $[0,1]$.

[*You may wish to use the identity* $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \ldots + b^{k-1})$.]

Suppose that A is a closed subalgebra of the algebra $C(X)$ of continuous real-valued functions on a compact metric space (X,d), equipped with the uniform norm, and suppose that A has the property that for each $x \in X$ there exists $a \in A$ with $a(x) \neq 0$. Show that there exists $h \in A$ such that $0 < h(x) \leqslant 1$ for all $x \in X$.

Show that $h^{1/k} \in A$ for each positive integer k, and show that A contains the constant functions.

4H Groups, Rings and Fields

(i) What does it mean for a ring to be Noetherian? State Hilbert's Basis Theorem. Give an example of a Noetherian ring which is not a principal ideal domain.

(ii) Prove Hilbert's Basis Theorem.

Is it true that if the ring $R[X]$ is Noetherian, then so is R?

5D Electromagnetism

(i) A plane electromagnetic wave in a vacuum has an electric field

$$\mathbf{E} = (E_1, E_2, 0)\cos(kz - \omega t),$$

referred to cartesian axes (x, y, z). Show that this wave is plane polarized and find the orientation of the plane of polarization. Obtain the corresponding plane polarized magnetic field and calculate the rate at which energy is transported by the wave.

(ii) Suppose instead that

$$\mathbf{E} = (E_1\cos(kz - \omega t),\ E_2\cos(kz - \omega t + \phi), 0),$$

with ϕ a constant, $0 < \phi < \pi$. Show that, if the axes are now rotated through an angle ψ so as to obtain an elliptically polarized wave with an electric field

$$\mathbf{E}' = (F_1\cos(kz - \omega t + \chi),\ F_2\sin(kz - \omega t + \chi), 0),$$

then

$$\tan 2\psi = \frac{2E_1E_2\cos\phi}{E_1^2 - E_2^2}.$$

Show also that if $E_1 = E_2 = E$ there is an elliptically polarized wave with

$$\mathbf{E}' = \sqrt{2}E\left(\cos(kz - \omega t + \tfrac{1}{2}\phi)\cos\tfrac{1}{2}\phi,\ \sin(kz - \omega t + \tfrac{1}{2}\phi)\sin\tfrac{1}{2}\phi, 0\right).$$

6F Dynamics of Differential Equations

(i) Define the Floquet multiplier and Liapunov exponent for a periodic orbit $\hat{\mathbf{x}}(t)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Show that one multiplier is always unity, and that the other is given by

$$\exp\left(\int_0^T \nabla\cdot \mathbf{f}(\hat{\mathbf{x}}(t))dt\right) \, , \qquad (*)$$

where T is the period of the orbit.

The Van der Pol oscillator $\ddot{x} + \epsilon\dot{x}(x^2 - 1) + x = 0\,,\; 0 < \epsilon \ll 1$ has a limit cycle $\hat{x}(t) \approx 2\sin t$. Show using $(*)$ that this orbit is stable.

(ii) Show, by considering the normal form for a Hopf bifurcation from a fixed point $\mathbf{x}_0(\mu)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$, that in some neighbourhood of the bifurcation the periodic orbit is stable when it exists in the range of μ for which $\mathbf{x}_0$ is unstable, and unstable in the opposite case.

Now consider the system

$$\left.\begin{aligned} \dot{x} &= x(1-y) + \mu x \\ \dot{y} &= y(x-1) - \mu x \end{aligned}\right\} \qquad x > 0\,.$$

Show that the fixed point $(1+\mu\,,\; 1+\mu)$ has a Hopf bifurcation when $\mu = 0$, and is unstable (stable) when $\mu > 0$ ($\mu < 0$).

Suppose that a periodic orbit exists in $\mu > 0$. Show without solving for the orbit that the result of part (i) shows that such an orbit is unstable. Define a similar result for $\mu < 0$.

What do you conclude about the existence of periodic orbits when $\mu \neq 0$? Check your answer by applying Dulac's criterion to the system, using the weighting $\rho = e^{-(x+y)}$.

7K Geometry of Surfaces

(i) State what it means for surfaces $f : U \to \mathbb{R}^3$ and $g : V \to \mathbb{R}^3$ to be isometric.

Let $f : U \to \mathbb{R}^3$ be a surface, $\phi : V \to U$ a diffeomorphism, and let $g = f \circ \phi : V \to \mathbb{R}^3$.

State a formula comparing the first fundamental forms of f and g.

(ii) Give a proof of the formula referred to at the end of part (i). Deduce that "isometry" is an equivalence relation.

The *catenoid* and the *helicoid* are the surfaces defined by

$$(u, v) \to (u\cos v,\; u\sin v,\; v)$$

and

$$(\vartheta, z) \to (\cosh z\cos\vartheta,\; \cosh z\sin\vartheta,\; z).$$

Show that the catenoid and the helicoid are isometric.

8J Logic, Computation and Set Theory

(i) Explain briefly what is meant by the terms *register machine* and *computable function.*

Let u be the universal computable function $u(m,n) = f_m(n)$ and s a total computable function with $f_{s(m,n)}(k) = f_m(n,k)$. Here $f_m(n)$ and $f_m(n,k)$ are the unary and binary functions computed by the m-th register machine program P_m. Suppose $h : \mathbb{N} \to \mathbb{N}$ is a total computable function. By considering the function

$$g(m,n) = u(h(s(m,m)),n)$$

show that there is a number a such that $f_a = f_{h(a)}$.

(ii) Let P be the set of all partial functions $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Consider the mapping $\Phi : P \to P$ defined by

$$\Phi(g)(m,n) = \begin{cases} n+1 & \text{if } m = 0, \\ g(m-1,1) & \text{if } m > 0,\ n = 0 \text{ and } g(m-1,1) \text{ defined,} \\ g(m-1,g(m,n-1)) & \text{if } mn > 0 \text{ and } g(m-1,g(m,n-1)) \text{ defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(a) Show that any fixed point of Φ is a total function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Deduce that Φ has a unique fixed point.
[*The Bourbaki-Witt Theorem may be assumed if stated precisely.*]

(b) It follows from standard closure properties of the computable functions that there is a computable function ψ such that

$$\psi(p,m,n) = \Phi(f_p)(m,n).$$

Assuming this, show that there is a total computable function h such that

$$\Phi(f_p) = f_{h(p)} \text{ for all } p.$$

Deduce that the fixed point of Φ is computable.

9J Number Theory

(i) Let $\pi(x)$ denote the number of primes $\leqslant x$, where x is a positive real number. State and prove Legendre's formula relating $\pi(x)$ to $\pi(\sqrt{x})$. Use this formula to compute $\pi(100) - \pi(10)$.

(ii) Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, where s is a real number greater than 1. Prove the following two assertions rigorously, assuming always that $s > 1$.

(a) $\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}$, where the product is taken over all primes p;

(b) $\zeta(s) = \dfrac{1}{1 - 2^{1-s}} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$.

Explain why (b) enables us to define $\zeta(s)$ for $0 < s < 1$. Deduce from (b) that $\lim_{s \to 1} (s-1)\zeta(s) = 1$.

10M Algorithms and Networks

(i) Consider the unconstrained geometric programme GP

$$\text{minimise} \quad g(t) = \sum_{i=1}^{n} c_i \, \Pi_{j=1}^{m} \, t_j^{a_{ij}}$$

$$\text{subject to} \quad t_j > 0 \quad j = 1, \ldots, m.$$

State the dual problem to GP. Give a careful statement of the AM-GM inequality, and use it to prove the primal-dual inequality for GP.

(ii) Define min-path and max-tension problems. State and outline the proof of the max-tension min-path theorem.

A company has branches in five cities A, B, C, D and E. The fares for direct flights between these cities are as follows:

	A	B	C	D	E
A	–	50	40	25	10
B	50	–	20	90	25
C	40	20	–	10	25
D	25	90	10	–	55
E	10	25	25	55	–

Formulate this as a min-path problem. Illustrate the max-tension min-path algorithm by finding the cost of travelling by the cheapest routes between D and each of the other cities.

11L Stochastic Financial Models

(i) Explain briefly what it means to say that a stochastic process $\{W_t,\ t \geqslant 0\}$ is a standard Brownian motion.

Let $\{W_t,\ t \geqslant 0\}$ be a standard Brownian motion and let a, b be real numbers. What condition must a and b satisfy to ensure that the process e^{aW_t+bt} is a martingale? Justify your answer carefully.

(ii) At the beginning of each of the years $r = 0, 1, \ldots, n-1$ an investor has income X_r, of which he invests a proportion ρ_r, $0 \leqslant \rho_r \leqslant 1$, and consumes the rest during the year. His income at the beginning of the next year is

$$X_{r+1} = X_r + \rho_r X_r W_r,$$

where $W_0, \ldots, W_{n-1}$ are independent positive random variables with finite means and $X_0 \geqslant 0$ is a constant. He decides on ρ_r after he has observed both X_r and W_r at the beginning of year r, but at that time he does not have any knowledge of the value of W_s, for any $s > r$. The investor retires in year n and consumes his entire income during that year. He wishes to determine the investment policy that maximizes his expected total consumption

$$\mathbb{E}\left[\sum_{r=0}^{n-1}(1-\rho_r)X_r + X_n\right].$$

Prove that the optimal policy may be expressed in terms of the numbers $b_0, b_1, \ldots,$ b_n where $b_n = 1$, $b_r = b_{r+1} + \mathbb{E}\max(b_{r+1}W_r, 1)$, for $r < n$, and determine the optimal expected total consumption.

12M Principles of Statistics

(i) Describe in detail how to perform the Wald, score and likelihood ratio tests of a *simple* null hypothesis $H_0 : \theta = \theta_0$ given a random sample $X_1, \ldots, X_n$ from a regular one-parameter density $f(x; \theta)$. In each case you should specify the asymptotic null distribution of the test statistic.

(ii) Let $X_1, \ldots, X_n$ be an independent, identically distributed sample from a distribution F, and let $\hat{\theta}(X_1, \ldots, X_n)$ be an estimator of a parameter θ of F.

Explain what is meant by: (a) the *empirical distribution function* of the sample; (b) the *bootstrap estimator* of the *bias* of $\hat{\theta}$, based on the empirical distribution function. Explain how a bootstrap estimator of the *distribution function* of $\hat{\theta} - \theta$ may be used to construct an approximate $1 - \alpha$ confidence interval for θ.

Suppose the parameter of interest is $\theta = \mu^2$, where μ is the mean of F, and the estimator is $\hat{\theta} = \bar{X}^2$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ is the sample mean.

Derive an *explicit* expression for the bootstrap estimator of the bias of $\hat{\theta}$ and show that it is biased as an estimator of the true bias of $\hat{\theta}$.

Let $\hat{\theta}_i$ be the value of the estimator $\hat{\theta}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ computed from the sample of size $n-1$ obtained by deleting X_i and let $\hat{\theta}_{.} = n^{-1} \sum_{i=1}^{n} \hat{\theta}_i$. The *jackknife* estimator of the bias of $\hat{\theta}$ is

$$b_J = (n-1)\ (\hat{\theta}_{.} - \hat{\theta})\ .$$

Derive the jackknife estimator b_J for the case $\hat{\theta} = \bar{X}^2$, and show that, as an estimator of the true bias of $\hat{\theta}$, it is unbiased.

13E Foundations of Quantum Mechanics

(i) Two particles with angular momenta j_1, j_2 and basis states $|j_1\ m_1\rangle, |j_2\ m_2\rangle$ are combined to give total angular momentum j and basis states $|j\ m\rangle$. State the possible values of j, m and show how a state with $j = m = j_1 + j_2$ can be constructed. Briefly describe, for a general allowed value of j, what the Clebsch-Gordan coefficients are.

(ii) If the angular momenta j_1 and j_2 are both 1 show that the combined state $|2\ 0\rangle$ is

$$|2\ 0\rangle = \sqrt{\frac{1}{6}}\Big(|1\ 1\rangle|1\ {-1}\rangle + |1\ {-1}\rangle|1\ 1\rangle\Big) + \sqrt{\frac{2}{3}}|1\ 0\rangle|1\ 0\rangle.$$

Determine the corresponding expressions for the combined states $|1\ 0\rangle$ and $|0\ 0\rangle$, assuming that they are respectively antisymmetric and symmetric under interchange of the two particles.

If the combined system is in state $|0\ 0\rangle$ what is the probability that measurements of the z-component of angular momentum for either constituent particle will give the value of 1?

[*Hint:* $J_{\pm}|j\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m \pm 1\rangle$.]

14D Statistical Physics and Cosmology

(i) Write down the first law of thermodynamics for the change dU in the internal energy $U(N, V, S)$ of a gas of N particles in a volume V with entropy S.

Given that

$$PV = (\gamma - 1)U,$$

where P is the pressure, use the first law to show that PV^γ is constant at constant N and S.

Write down the Boyle-Charles law for a non-relativistic ideal gas and hence deduce that the temperature T is proportional to $V^{1-\gamma}$ at constant N and S.

State the principle of equipartition of energy and use it to deduce that

$$U = \frac{3}{2}NkT.$$

Hence deduce the value of γ. Show that this value of γ is such that the ratio E_i/kT is unchanged by a change of volume at constant N and S, where E_i is the energy of the i-th one particle eigenstate of a non-relativistic ideal gas.

(ii) A classical gas of non-relativistic particles of mass m at absolute temperature T and number density n has a chemical potential

$$\mu = mc^2 - kT \ln\left(\frac{g_s}{n}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right),$$

where g_s is the particle's spin degeneracy factor. What condition on n is needed for the validity of this formula and why?

Thermal and chemical equilibrium between two species of non-relativistic particles a and b is maintained by the reaction

$$a + \alpha \leftrightarrow b + \beta,$$

where α and β are massless particles with zero chemical potential. Given that particles a and b have masses m_a and m_b respectively, but equal spin degeneracy factors, find the number density ratio n_a/n_b as a function of m_a, m_b and T. Given that $m_a > m_b$ but $m_a - m_b \ll m_b$ show that

$$\frac{n_a}{n_b} \approx f\left(\frac{(m_a - m_b)c^2}{kT}\right)$$

for some function f which you should determine.

Explain how a reaction of the above type is relevant to a determination of the neutron to proton ratio in the early universe and why this ratio does not fall rapidly to zero as the universe cools. Explain briefly the process of primordial nucleosynthesis by which neutrons are converted into stable helium nuclei. Let

$$Y_{He} = \frac{\rho_{He}}{\rho}$$

be the fraction of the universe that ends up in helium. Compute Y_{He} as a function of the ratio $r = n_a/n_b$ at the time of nucleosynthesis.

15E Symmetries and Groups in Physics

(i) Let D_6 denote the symmetry group of rotations and reflections of a regular hexagon. The elements of D_6 are given by $\{e, c, c^2, c^3, c^4, c^5, b, bc, bc^2, bc^3, bc^4, bc^5\}$ with $c^6 = b^2 = e$ and $cb = bc^5$. The conjugacy classes of D_6 are $\{e\}$, $\{c, c^5\}$, $\{c^2, c^4\}$, $\{c^3\}$, $\{b, bc^2, bc^4\}$ and $\{bc, bc^3, bc^5\}$.

Show that the character table of D_6 is

D_6	e	$\{c, c^5\}$	$\{c^2, c^4\}$	$\{c^3\}$	$\{b, bc^2, bc^4\}$	$\{bc, bc^3, bc^5\}$
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2	1	-1	-2	0	0
χ_6	2	-1	-1	2	0	0

(ii) Show that the character of an $SO(3)$ rotation with angle θ in the $2l+1$ dimensional irreducible representation of $SO(3)$ is given by

$$\chi_l(\theta) = 1 + 2\cos\theta + 2\cos(2\theta) + \ldots + 2\cos((l-1)\theta) + 2\cos(l\theta)\ .$$

For a hexagonal crystal of atoms find how the degeneracy of the D-wave orbital states ($l = 2$) in the atomic central potential is split by the crystal potential with D_6 symmetry and give the new degeneracies.

By using the fact that D_3 is isomorphic to $D_6/\{e, c^3\}$, or otherwise, find the degeneracies of eigenstates if the hexagonal symmetry is broken to the subgroup D_3 by a deformation. The introduction of a magnetic field further reduces the symmetry to C_3. What will the degeneracies of the energy eigenstates be now?

16C Transport Processes

(i) A layer of fluid of depth $h(x,t)$, density ρ and viscosity μ sits on top of a rigid horizontal plane at $y=0$. Gravity g acts vertically and surface tension is negligible.

Assuming that the horizontal velocity component u and pressure p satisfy the lubrication equations

$$\begin{aligned} 0 &= -p_x + \mu u_{yy} \\ 0 &= -p_y - \rho g, \end{aligned}$$

together with appropriate boundary conditions at $y=0$ and $y=h$ (which should be stated), show that h satisfies the partial differential equation

$$h_t = \frac{g}{3\nu}\left(h^3 h_x\right)_x, \qquad (*)$$

where $\nu = \mu/\rho$.

(ii) A two-dimensional blob of the above fluid has fixed area A and time-varying width $2X(t)$, such that

$$A = \int_{-X(t)}^{X(t)} h(x,t)\,dx.$$

The blob spreads under gravity.

Use scaling arguments to show that, after an initial transient, $X(t)$ is proportional to $t^{1/5}$ and $h(0,t)$ is proportional to $t^{-1/5}$. Hence show that equation $(*)$ of Part (i) has a similarity solution of the form

$$h(x,t) = \left(\frac{A^2\nu}{gt}\right)^{1/5} H(\xi), \quad \text{where} \quad \xi = \frac{x}{\left(A^3 gt/\nu\right)^{1/5}},$$

and find the differential equation satisfied by $H(\xi)$.

Deduce that

$$H = \begin{cases} \left[\frac{9}{10}\left(\xi_0^2 - \xi^2\right)\right]^{1/3} & \text{in} \quad -\xi_0 < \xi < \xi_0 \\ 0 & \text{in} \quad |\xi| > \xi_0 \ , \end{cases}$$

where

$$X(t) = \xi_0 \left(\frac{A^3 gt}{\nu}\right)^{1/5}.$$

Express ξ_0 in terms of the integral

$$I = \int_{-1}^{1} \left(1-u^2\right)^{1/3} du.$$

17C Mathematical Methods

(i) State the Fredholm alternative for Fredholm integral equations of the second kind.

Show that the integral equation

$$\phi(x) - \lambda \int_0^1 (x+t)\phi(t)dt = f(x)\,, \quad 0 \leqslant x \leqslant 1\ ,$$

where f is a continuous function, has a unique solution for ϕ if $\lambda \neq -6 \pm 4\sqrt{3}$. Derive this solution.

(ii) Describe the WKB method for finding approximate solutions $f(x)$ of the equation

$$\frac{d^2 f(x)}{dx^2} + q(\epsilon x) f(x) = 0\ ,$$

where q is an arbitrary non-zero, differentiable function and ϵ is a small parameter. Obtain these solutions in terms of an exponential with slowly varying exponent and slowly varying amplitude.

Hence, by means of a suitable change of independent variable, find approximate solutions $w(t)$ of the equation

$$\frac{d^2 w}{dt^2} + \lambda^2 t w = 0\ ,$$

in $t > 0$, where λ is a large parameter.

18F Nonlinear Waves

(i) Show that the equation

$$\frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi}{\partial x^2} + 1 - \phi^2 = 0$$

has two solutions which are independent of both x and t. Show that one of these is linearly stable. Show that the other solution is linearly unstable, and find the range of wavenumbers that exhibit the instability.

Sketch the nonlinear evolution of the unstable solution after it receives a small, smooth, localized perturbation in the direction towards the stable solution.

(ii) Show that the equations

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} &= e^{-u+v} \quad , \\ -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} &= e^{-u-v} \end{aligned}$$

are a Bäcklund pair for the equations

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-2u} \, , \quad \frac{\partial^2 v}{\partial x \partial y} = 0 \, .$$

By choosing v to be a suitable constant, and using the Bäcklund pair, find a solution of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-2u}$$

which is non-singular in the region $y < 4x$ of the (x, y) plane and has the value $u = 0$ at $x = \frac{1}{2}$, $y = 0$.

19F Numerical Analysis

(i) Determine the order of the multistep method

$$\mathbf{y}_{n+2} - (1+\alpha)\mathbf{y}_{n+1} + \alpha \mathbf{y}_n = h[\tfrac{1}{12}(5+\alpha)\mathbf{f}_{n+2} + \tfrac{2}{3}(1-\alpha)\mathbf{f}_{n+1} - \tfrac{1}{12}(1+5\alpha)\mathbf{f}_n]$$

for the solution of ordinary differential equations for different choices of α in the range $-1 \leqslant \alpha \leqslant 1$.

(ii) Prove that no such choice of α results in a method whose linear stability domain includes the interval $(-\infty, \ 0)$.