UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Tuesday 4 June 2002 9 to 12

PAPER 2

Before you begin read these instructions carefully.

Each question is divided into Part ***(i)*** *and Part* ***(ii)****, which may or may not be related. Candidates may attempt either or both Parts of any question, but must not attempt Parts from more than* ***SIX*** *questions. If you submit answers to Parts of more than six questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same, with Part ***(ii)*** *of each question carrying twice as many marks as Part* ***(i)****. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***3K, 7K*** *should be in one bundle and* ***1M, 10M*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *Parts of* ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) In each of the following cases, the state-space I and non-zero transition rates q_{ij} $(i \neq j)$ of a continuous-time Markov chain are given. Determine in which cases the chain is explosive.

(a) $I = \{1, 2, 3, \ldots\}$, $q_{i,i+1} = i^2$, $i \in I$,
(b) $I = \mathbb{Z}$, $q_{i,i-1} = q_{i,i+1} = 2^i$, $i \in I$.

(ii) Children arrive at a see-saw according to a Poisson process of rate 1. Initially there are no children. The first child to arrive waits at the see-saw. When the second child arrives, they play on the see-saw. When the third child arrives, they all decide to go and play on the merry-go-round. The cycle then repeats. Show that the number of children at the see-saw evolves as a Markov Chain and determine its generator matrix. Find the probability that there are no children at the see-saw at time t.

Hence obtain the identity

$$\sum_{n=0}^{\infty} e^{-t} \frac{t^{3n}}{(3n)!} = \frac{1}{3} + \frac{2}{3}\, e^{-\frac{3}{2}t} \cos \frac{\sqrt{3}}{2} t \ .$$

2G Principles of Dynamics

(i) A number N of non-interacting particles move in one dimension in a potential $V(x,t)$. Write down the Hamiltonian and Hamilton's equations for one particle.

At time t, the number density of particles in phase space is $f(x,p,t)$. Write down the time derivative of f along a particle's trajectory. By equating the rate of change of the number of particles in a fixed domain V in phase space to the flux into V across its boundary, deduce that f is a constant along any particle's trajectory.

(ii) Suppose that $V(x) = \frac{1}{2} m\omega^2 x^2$, and particles are injected in such a manner that the phase space density is a constant f_1 at any point of phase space corresponding to a particle energy being smaller than E_1 and zero elsewhere. How many particles are present?

Suppose now that the potential is very slowly altered to the square well form

$$V(x) = \begin{cases} 0, & -L < x < L \\ \infty & \text{elsewhere} . \end{cases}$$

Show that the greatest particle energy is now

$$E_2 = \frac{\pi^2}{8} \frac{E_1^2}{mL^2\omega^2}.$$

3K Functional Analysis

(i) State and prove the parallelogram law for Hilbert spaces.

Suppose that K is a closed linear subspace of a Hilbert space H and that $x \in H$. Show that x is orthogonal to K if and only if 0 is the nearest point to x in K.

(ii) Suppose that H is a Hilbert space and that ϕ is a continuous linear functional on H with $\|\phi\| = 1$. Show that there is a sequence (h_n) of unit vectors in H with $\phi(h_n)$ real and $\phi(h_n) > 1 - 1/n$.

Show that h_n converges to a unit vector h, and that $\phi(h) = 1$.

Show that h is orthogonal to N, the null space of ϕ, and also that $H = N \oplus \mathrm{span}(h)$.

Show that $\phi(k) = \langle k, h \rangle$, for all $k \in H$.

4H Groups, Rings and Fields

(i) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(ii) What are the units in $\mathbb{Z}[i]$? What are the primes in $\mathbb{Z}[i]$? Justify your answers.

Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

5D Electromagnetism

(i) Show that the Lorentz force corresponds to a curvature force and the gradient of a magnetic pressure, and that it can be written as the divergence of a second rank tensor, the Maxwell stress tensor.

Consider the potential field $\mathbf{B}$ given by $\mathbf{B} = -\nabla\Phi$, where

$$\Phi(x, y) = \left(\frac{B_0}{k}\right) \cos kx \; e^{-ky},$$

referred to cartesian coordinates (x, y, z). Obtain the Maxwell stress tensor and verify that its divergence vanishes.

(ii) The magnetic field in a stellar atmosphere is maintained by steady currents and the Lorentz force vanishes. Show that there is a scalar field α such that $\nabla \wedge \mathbf{B} = \alpha\mathbf{B}$ and $\mathbf{B} \cdot \nabla\alpha = 0$. Show further that if α is constant, then $\nabla^2\mathbf{B} + \alpha^2\mathbf{B} = 0$. Obtain a solution in the form $\mathbf{B} = (B_1(z), B_2(z), 0)$; describe the structure of this field and sketch its variation in the z-direction.

6F Dynamics of Differential Equations

(i) Define the terms *stable manifold* and *unstable manifold* of a hyperbolic fixed point $\mathbf{x}_0$ of a dynamical system. State carefully the stable manifold theorem.

Give an approximation, correct to fourth order in $|\mathbf{x}|$, for the stable and unstable manifolds of the origin for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x + x^2 - y^2 \\ -y + x^2 \end{pmatrix} .$$

(ii) State, without proof, the centre manifold theorem. Show that the fixed point at the origin of the system

$$\begin{aligned} \dot{x} &= y - x + ax^3 \ , \\ \dot{y} &= rx - y - zy \ , \\ \dot{z} &= -z + xy \ , \end{aligned}$$

where a is a constant, is non-hyperbolic at $r = 1$.

Using new coordinates $v = x + y\ ,\ w = x - y$, find the centre manifold in the form

$$w = \alpha v^3 + \dots \ , \quad z = \beta v^2 + \gamma v^4 + \dots$$

for constants α, β, γ to be determined. Hence find the evolution equation on the centre manifold in the form

$$\dot{v} = \frac{1}{8}(a-1)v^3 + \left(\frac{(3a+1)(a+1)}{128} + \frac{(a-1)}{32} \right) v^5 + \dots \ .$$

Ignoring higher order terms, give conditions on a that guarantee that the origin is asymptotically stable.

7K Geometry of Surfaces

(i)

Consider the surface

$$z = \frac{1}{2}(\lambda x^2 + \mu y^2) + h(x, y),$$

where $h(x, y)$ is a term of order at least 3 in x, y. Calculate the first fundamental form at $x = y = 0$.

(ii) Calculate the second fundamental form, at $x = y = 0$, of the surface given in Part (i). Calculate the Gaussian curvature. Explain why your answer is consistent with Gauss' "Theorema Egregium".

8H Graph Theory

(i) Define the chromatic polynomial $p(G\,;t)$ of the graph G, and establish the standard identity

$$p(G\,;t) \;=\; p(G-e\,;t) \;-\; p(G/e\,;t),$$

where e is an edge of G. Deduce that, if G has n vertices and m edges, then

$$p(G\,;t) = a_n t^n - a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + (-1)^n a_0\,,$$

where $a_n = 1$, $a_{n-1} = m$ and $a_j \geq 0$ for $0 \leq j \leq n$.

(ii) Let G and $p(G\,;t)$ be as in Part (i). Show that if G has k components $G_1, \ldots, G_k$ then $p(G\,;t) \;=\; \prod_{i=1}^{k} p(G_i\,;t)$. Deduce that $a_k > 0$ and $a_j = 0$ for $0 \leq j < k$.

Show that if G is a tree then $p(G\,;t) = t(t-1)^{n-1}$. Must the converse hold? Justify your answer.

Show that if $p(G\,;t) = p(T_r(n)\,;t)$, where $T_r(n)$ is a Turán graph, then $G = T_r(n)$.

9H Coding and Cryptography

(i) Explain the idea of public key cryptography. Give an example of a public key system, explaining how it works.

(ii) What is a general feedback register of length d with initial fill $(X_0, \ldots, X_{d-1})$? What is the maximal period of such a register, and why? What does it mean for such a register to be linear?

Describe and justify the Berlekamp-Massey algorithm for breaking a cypher stream arising from a general linear feedback register of unknown length.

Use the Berlekamp-Massey algorithm to find a linear recurrence in $\mathbb{F}_2$ with first eight terms 1, 1, 0, 0, 1, 0, 1, 1.

10M Algorithms and Networks

(i) Let G be a directed network with nodes N, arcs A and capacities specified on each of the arcs. Define the terms *feasible flow*, *divergence, cut, upper* and *lower cut capacities.* Given two disjoint sets of nodes N^+ and N^-, what does it mean to say that a cut Q separates N^+ from N^-? Prove that the flux of a feasible flow x from N^+ to N^- is bounded above by the upper capacity of Q, for any cut Q separating N^+ from N^-.

(ii) Define the maximum-flow and minimum-cut problems. State the max-flow min-cut theorem and outline the main steps of the maximum-flow algorithm. Use the algorithm to find the maximum flow between the nodes 1 and 5 in a network whose node set is $\{1, 2, \ldots, 5\}$, where the lower capacity of each arc is 0 and the upper capacity c_{ij} of the directed arc joining node i to node j is given by the (i, j)-entry in the matrix

$$\begin{pmatrix} 0 & 7 & 9 & 8 & 0 \\ 0 & 0 & 6 & 8 & 4 \\ 0 & 9 & 0 & 2 & 10 \\ 0 & 3 & 7 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

[*The painted-network theorem can be used without proof but should be stated clearly. You may assume in your description of the maximum-flow algorithm that you are given an initial feasible flow.*]

11M Principles of Statistics

(i) Let X be a random variable with density function $f(x; \theta)$. Consider testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1$.

What is the form of the optimal size α classical hypothesis test?

Compare the form of the test with the Bayesian test based on the Bayes factor, *and* with the Bayes decision rule under the 0-1 loss function, under which a loss of 1 is incurred for an incorrect decision and a loss of 0 is incurred for a correct decision.

(ii) What does it mean to say that a family of densities $\{f(x; \theta), \theta \in \Theta\}$ with real scalar parameter θ is of *monotone likelihood ratio*?

Suppose X has a distribution from a family which is of monotone likelihood ratio with respect to a statistic $t(X)$ and that it is required to test $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$.

State, without proof, a theorem which establishes the existence of a uniformly most powerful test and describe in detail the form of the test.

Let $X_1, \ldots, X_n$ be independent, identically distributed $U(0, \theta)$, $\theta > 0$. Find a uniformly most powerful size α test of $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$, and find its power function. Show that we may construct a different, randomised, size α test with the same power function for $\theta \geqslant \theta_0$.

12L Computational Statistics and Statistical Modelling

(i) Suppose that the random variable Y has density function of the form

$$f(y|\theta,\phi) = \exp\left[\frac{y\theta - b(\theta)}{\phi} + c(y,\phi)\right]$$

where $\phi > 0$. Show that Y has expectation $b'(\theta)$ and variance $\phi b''(\theta)$.

(ii) Suppose now that $Y_1,\ldots,Y_n$ are independent negative exponential variables, with Y_i having density function $f(y_i|\mu_i) = \frac{1}{\mu_i}e^{-y_i/\mu_i}$ for $y_i > 0$. Suppose further that $g(\mu_i) = \beta^T x_i$ for $1 \leqslant i \leqslant n$, where $g(\cdot)$ is a known 'link' function, and $x_1,\ldots,x_n$ are given covariate vectors, each of dimension p. Discuss carefully the problem of finding $\hat{\beta}$, the maximum-likelihood estimator of β, firstly for the case $g(\mu_i) = 1/\mu_i$, and secondly for the case $g(\mu) = \log\mu_i$; in both cases you should state the large-sample distribution of $\hat{\beta}$.

[*Any standard theorems used need not be proved.*]

13E Foundations of Quantum Mechanics

(i) A Hamiltonian H_0 has energy eigenvalues E_r and corresponding non-degenerate eigenstates $|r\rangle$. Show that under a small change in the Hamiltonian δH,

$$\delta|r\rangle = \sum_{s\neq r}\frac{\langle s|\delta H|r\rangle}{E_r - E_s}|s\rangle,$$

and derive the related formula for the change in the energy eigenvalue E_r to first and second order in δH.

(ii) The Hamiltonian for a particle moving in one dimension is $H = H_0 + \lambda H'$, where $H_0 = p^2/2m + V(x)$, $H' = p/m$ and λ is small. Show that

$$\frac{i}{\hbar}[H_0, x] = H'$$

and hence that

$$\delta E_r = -\lambda^2\frac{i}{\hbar}\langle r|H'x|r\rangle = \lambda^2\frac{i}{\hbar}\langle r|xH'|r\rangle$$

to second order in λ.

Deduce that δE_r is independent of the particular state $|r\rangle$ and explain why this change in energy is exact to all orders in λ.

14E Quantum Physics

(i) A simple model of a one-dimensional crystal consists of an infinite array of sites equally spaced with separation a . An electron occupies the nth site with a probability amplitude c_n . The time-dependent Schrödinger equation governing these amplitudes is

$$i\hbar\frac{dc_n}{dt} = E_0 c_n - A(c_{n-1} + c_{n+1}) \quad ,$$

where E_0 is the energy of an electron at an isolated site and the amplitude for transition between neighbouring sites is $A > 0$. By examining a solution of the form

$$c_n = e^{ikan - iEt/\hbar} \quad ,$$

show that E, the energy of the electron in the crystal, lies in a band

$$E_0 - 2A \leq E \leq E_0 + 2A \ .$$

Identify the Brillouin zone for this model and explain its significance.

(ii) In the above model the electron is now subject to an electric field $\mathcal{E}$ in the direction of increasing n . Given that the charge on the electron is $-e$ write down the new form of the time-dependent Schrödinger equation for the probability amplitudes. Show that it has a solution of the form

$$c_n = \exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon(t')dt' + i(k - \frac{e\mathcal{E}t}{\hbar})na\right\} \quad ,$$

where

$$\epsilon(t) = E_0 - 2A\cos\left((k - \frac{e\mathcal{E}t}{\hbar})a\right) \quad .$$

Explain briefly how to interpret this result and use it to show that the dynamical behaviour of an electron near the bottom of the energy band is the same as that for a free particle in the presence of an electric field with an effective mass $m^* = \hbar^2/(2Aa^2)$.

15D General Relativity

(i) Consider the line element describing the interior of a star,

$$ds^2 = e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{2\gamma(r)}dt^2 \ ,$$

defined for $0 \leq r \leq r_0$ by

$$e^{-2\alpha(r)} = 1 - Ar^2$$

and

$$e^{\gamma(r)} = \frac{3}{2}e^{-\alpha_0} - \frac{1}{2}e^{-\alpha(r)}.$$

Here $A = 2M/r_0^3$, M is the mass of the star, and α_0 is defined to be $\alpha(r_0)$.

The star is made of a perfect fluid with energy-momentum tensor

$$T_{ab} = (p + \rho)u_a u_b + p g_{ab}.$$

Here u^a is the 4-velocity of the fluid which is at rest, the density ρ is constant throughout the star ($0 \leq r \leq r_0$) and the pressure $p = p(r)$ depends only on the radial coordinate. Write down the Einstein field equations and show that (in geometrical units with $G = c = 1$) they may equivalently be written as

$$R_{ab} = 8\pi(p + \rho)u_a u_b + 4\pi(p - \rho)g_{ab}.$$

(ii) Using the formulae below, or otherwise, show that for $0 \leq r \leq r_0$ one has

$$\rho = \frac{3A}{8\pi}, \quad p(r) = \frac{3A}{8\pi}\left(\frac{e^{-\alpha(r)} - e^{-\alpha_0}}{3e^{-\alpha_0} - e^{-\alpha(r)}}\right).$$

[*The non-zero components of the Ricci tensor are:*

$$R_{11} = -\gamma'' + \alpha'\gamma' - \gamma'^2 + \frac{2\alpha'}{r}, \quad R_{22} = e^{-2\alpha}[(\alpha' - \gamma')r - 1] + 1,$$

$$R_{33} = \sin^2\theta R_{22}, \quad R_{44} = e^{2\gamma-2\alpha}[\gamma'' - \alpha'\gamma' + \gamma'^2 + \frac{2\gamma'}{r}].$$

Note that

$$\alpha' = Are^{2\alpha}, \quad \gamma' = \frac{1}{2}Are^{\alpha-\gamma}, \quad \gamma'' = \frac{1}{2}Ae^{\alpha-\gamma} + \frac{1}{2}A^2r^2e^{3\alpha-\gamma} - \frac{1}{4}A^2r^2e^{2\alpha-2\gamma}. \quad]$$

16G Theoretical Geophysics

(i) State the equations that relate strain to displacement and stress to strain in a linear, isotropic elastic solid.

In the absence of body forces, the Euler equation for infinitesimal deformations of a solid of density ρ is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} \quad .$$

Derive an equation for $\mathbf{u}(\mathbf{x}, t)$ in a linear, isotropic, homogeneous elastic solid. Hence show that both the dilatation $\theta = \boldsymbol{\nabla} \cdot \mathbf{u}$ and the rotation $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ satisfy wave equations and find the corresponding wave speeds α and β.

(ii) The ray parameter $p = r \sin i / v$ is constant along seismic rays in a spherically symmetric Earth, where $v(r)$ is the relevant wave speed (α or β) and $i(r)$ is the angle between the ray and the local radial direction.

Express $\tan i$ and $\sec i$ in terms of p and the variable $\eta(r) = r/v$. Hence show that the angular distance and travel time between a surface source and receiver, both at radius R, are given by

$$\Delta(p) = 2 \int_{r_m}^{R} \frac{p}{r} \frac{dr}{(\eta^2 - p^2)^{1/2}} \quad , \qquad T(p) = 2 \int_{r_m}^{R} \frac{\eta^2}{r} \frac{dr}{(\eta^2 - p^2)^{1/2}} \quad ,$$

where r_m is the minimum radius attained by the ray. What is $\eta(r_m)$?

A simple Earth model has a solid mantle in $R/2 < r < R$ and a liquid core in $r < R/2$. If $\alpha(r) = A/r$ in the mantle, where A is a constant, find $\Delta(p)$ and $T(p)$ for P-arrivals (direct paths lying entirely in the mantle), and show that

$$T = \frac{R^2 \sin \Delta}{A} \quad .$$

[*You may assume that* $\displaystyle\int \frac{du}{u\sqrt{u-1}} = 2\cos^{-1}\left(\frac{1}{\sqrt{u}}\right)$.]

Sketch the $T - \Delta$ curves for P and PcP arrivals on the same diagram and explain briefly why they terminate at $\Delta = \cos^{-1} \frac{1}{4}$.

17C Mathematical Methods

(i) Show that the equation

$$\epsilon x^4 - x^2 + 5x - 6 = 0\,, \quad |\epsilon| \ll 1\,,$$

has roots in the neighbourhood of $x = 2$ and $x = 3$. Find the first two terms of an expansion in ϵ for each of these roots.

Find a suitable series expansion for the other two roots and calculate the first two terms in each case.

(ii) Describe, giving reasons for the steps taken, how the leading-order approximation for $\lambda \gg 1$ to an integral of the form

$$I(\lambda) \equiv \int_A^B f(t) e^{i\lambda g(t)} dt\,,$$

where λ and g are real, may be found by the method of stationary phase. Consider the cases where (a) $g'(t)$ has one simple zero at $t = t_0$ with $A < t_0 < B$; (b) $g'(t)$ has more than one simple zero in $A < t < B$; and (c) $g'(t)$ has only a simple zero at $t = B$. What is the order of magnitude of $I(\lambda)$ if $g'(t)$ is non-zero for $A \le t \le B$?

Use the method of stationary phase to find the leading-order approximation to

$$J(\lambda) \equiv \int_0^1 \sin[\lambda(2t^4 - t)]\, dt$$

for $\lambda \gg 1$.

[*You may use the fact that* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2} du = \sqrt{\pi} e^{i\pi/4}$.]

18F Nonlinear Waves

(i) Find a travelling wave solution of unchanging shape for the modified Burgers equation (with $\alpha > 0$)

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$$

with $u = 0$ far ahead of the wave and $u = 1$ far behind. What is the velocity of the wave? Sketch the shape of the wave.

(ii) Explain why the method of characteristics, when applied to an equation of the type

$$\frac{\partial u}{\partial t} + c(u) \frac{\partial u}{\partial x} = 0\,,$$

with initial data $u(x,0) = f(x)$, sometimes gives a multi-valued solution. State the shock-fitting algorithm that gives a single-valued solution, and explain how it is justified.

Consider the equation above, with $c(u) = u^2$. Suppose that

$$u(x,0) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases} \quad .$$

Sketch the characteristics in the (x,t) plane. Show that a shock forms immediately, and calculate the velocity at which it moves.

19F Numerical Analysis

(i)

Given the finite-difference method

$$\sum_{k=-r}^{s} \alpha_k u_{m+k}^{n+1} = \sum_{k=-r}^{s} \beta_k u_{m+k}^{n}, \quad m,n \in \mathbb{Z},\ n \geqslant 0,$$

define

$$H(z) = \frac{\sum\limits_{k=-r}^{s} \beta_k z^k}{\sum\limits_{k=-r}^{s} \alpha_k z^k}.$$

Prove that this method is stable if and only if

$$|H(e^{i\theta})| \leqslant 1, \quad -\pi \leqslant \theta \leqslant \pi.$$

[*You may quote without proof known properties of the Fourier transform.*]

(ii) Find the range of the parameter μ such that the method

$$(1-2\mu)u_{m-1}^{n+1} + 4\mu u_m^{n+1} + (1-2\mu)u_{m+1}^{n+1} = u_{m-1}^{n} + u_{m+1}^{n}$$

is stable. Supposing that this method is used to solve the diffusion equation for $u(x,t)$, determine the order of magnitude of the local error as a power of Δx.