UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 7 June 2002 9 to 12

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, …, H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1E Analysis II

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f = (u, v)$, where u and v are defined by $u(0) = v(0) = 0$ and, for $t \neq 0$, $u(t) = t^2 \sin(1/t)$ and $v(t) = t^2 \cos(1/t)$. Show that f is differentiable on $\mathbb{R}$.

Show that for any real non-zero a, $||f'(a) - f'(0)|| > 1$, where we regard $f'(a)$ as the vector $\big(u'(a), v'(a)\big)$ in $\mathbb{R}^2$.

2A Methods

Write down the wave equation for the displacement $y(x, t)$ of a stretched string with constant mass density and tension. Obtain the general solution in the form

$$y(x, t) = f(x + ct) + g(x - ct),$$

where c is the wave velocity. For a solution in the region $0 \leqslant x < \infty$, with $y(0, t) = 0$ and $y \to 0$ as $x \to \infty$, show that

$$E = \int_0^\infty \left[\frac{1}{2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} c^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx,$$

is constant in time. Express E in terms of the general solution in this case.

3G Further Analysis

Let $f : X \to Y$ be a continuous map between topological spaces. Let

$$G_f = \{(x, f(x)) : \ x \in X\}.$$

(a) Show that if Y is Hausdorff, then G_f is closed in $X \times Y$.

(b) Show that if X is compact, then G_f is also compact.

4E Geometry

State Euler's formula for a graph $\mathcal{G}$ with F faces, E edges and V vertices on the surface of a sphere.

Suppose that every face in $\mathcal{G}$ has at least three edges, and that at least three edges meet at every vertex of $\mathcal{G}$. Let F_n be the number of faces of $\mathcal{G}$ that have exactly n edges ($n \geqslant 3$), and let V_m be the number of vertices at which exactly m edges meet ($m \geqslant 3$). By expressing $6F - \sum_n nF_n$ in terms of the V_j, or otherwise, show that every convex polyhedron has at least four faces each of which is a triangle, a quadrilateral or a pentagon.

5H Optimization

Consider a two-person zero-sum game with a payoff matrix

$$\begin{pmatrix} 3 & b \\ 5 & 2 \end{pmatrix},$$

where $0 < b < \infty$. Here, the (i, j) entry of the matrix indicates the payoff to player one if he chooses move i and player two move j. Suppose player one chooses moves 1 and 2 with probabilities p and $1 - p$, $0 \leq p \leq 1$. Write down the maximization problem for the optimal strategy and solve it for each value of b.

6B Numerical Analysis

For numerical integration, a quadrature formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

is applied which is exact on $\mathcal{P}_n$, i.e., for all polynomials of degree n.

Prove that such a formula is exact for all $f \in \mathcal{P}_{2n+1}$ if and only if x_i, $i = 0, \ldots, n$, are the zeros of an orthogonal polynomial $p_{n+1} \in \mathcal{P}_{n+1}$ which satisfies $\int_a^b p_{n+1}(x) r(x)\,dx = 0$ for all $r \in \mathcal{P}_n$. [*You may assume that* p_{n+1} *has* $(n+1)$ *distinct zeros.*]

7F Linear Mathematics

Which of the following statements are true, and which false? Give brief justifications for your answers.

(a) If U and W are subspaces of a vector space V, then $U \cap W$ is always a subspace of V.

(b) If U and W are distinct subspaces of a vector space V, then $U \cup W$ is never a subspace of V.

(c) If U, W and X are subspaces of a vector space V, then $U \cap (W + X) = (U \cap W) + (U \cap X)$.

(d) If U is a subspace of a finite-dimensional space V, then there exists a subspace W such that $U \cap W = \{0\}$ and $U + W = V$.

8C Fluid Dynamics

State and prove Kelvin's circulation theorem.

Consider a planar flow in the unbounded region outside a cylinder for which the vorticity vanishes everywhere at time $t = 0$. What may be deduced about the circulation around closed loops in the fluid at time t:

(i) that do not enclose the cylinder;

(ii) that enclose the cylinder?

Give a brief justification for your answer in each case.

9F Quadratic Mathematics

Explain what is meant by a *quadratic residue* modulo an odd prime p, and show that a is a quadratic residue modulo p if and only if $a^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}$. Hence characterize the odd primes p for which -1 is a quadratic residue.

State the law of quadratic reciprocity, and use it to determine whether 73 is a quadratic residue (mod 127).

10D Special Relativity

Write down the formulae for a Lorentz transformation with velocity v taking one set of co-ordinates (t, x) to another (t', x').

Imagine you observe a train travelling past Cambridge station at a relativistic speed u. Someone standing still on the train throws a ball in the direction the train is moving, with speed v. How fast do you observe the ball to be moving? Justify your answer.

SECTION II

11E Analysis II

Show that if a, b and c are non-negative numbers, and $a \leqslant b + c$, then

$$\frac{a}{1+a} \leqslant \frac{b}{1+b} + \frac{c}{1+c}.$$

Deduce that if (X, d) is a metric space, then $d(x,y)/[1 + d(x,y)]$ is a metric on X.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K_n = \{z \in D : |z| \leqslant (n-1)/n\}$. Let $\mathcal{F}$ be the class of continuous complex-valued functions on D and, for f and g in $\mathcal{F}$, define

$$\sigma(f,g) = \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{||f-g||_n}{1 + ||f-g||_n},$$

where $||f-g||_n = \sup\{|f(z) - g(z)| : z \in K_n\}$. Show that the series for $\sigma(f,g)$ converges, and that σ is a metric on $\mathcal{F}$.

For $|z| < 1$, let $s_k(z) = 1 + z + z^2 + \cdots + z^k$ and $s(z) = 1 + z + z^2 + \cdots$. Show that for $n \geqslant 2$, $||s_k - s||_n = n(1 - \frac{1}{n})^{k+1}$. By considering the sums for $2 \leqslant n \leqslant N$ and $n > N$ separately, show that for each N,

$$\sigma(s_k, s) \leqslant \sum_{n=2}^{N} ||s_k - s||_n + 2^{-N},$$

and deduce that $\sigma(s_k, s) \to 0$ as $k \to \infty$.

12A Methods

Consider the real Sturm-Liouville problem

$$\mathcal{L}y(x) = -(p(x)y')' + q(x)y = \lambda r(x)y,$$

with the boundary conditions $y(a) = y(b) = 0$, where p, q and r are continuous and positive on $[a, b]$. Show that, with suitable choices of inner product and normalisation, the eigenfunctions $y_n(x)$, $n = 1, 2, 3 \ldots$, form an orthonormal set.

Hence show that the corresponding Green's function $G(x, \xi)$ satisfying

$$(\mathcal{L} - \mu r(x))G(x, \xi) = \delta(x - \xi),$$

where μ is not an eigenvalue, is

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)}{\lambda_n - \mu},$$

where λ_n is the eigenvalue corresponding to y_n.

Find the Green's function in the case where

$$\mathcal{L}y \equiv y'',$$

with boundary conditions $y(0) = y(\pi) = 0$, and deduce, by suitable choice of μ, that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

13G Further Analysis

(a) Let f and g be two analytic functions on a domain D and let $\gamma \subset D$ be a simple closed curve homotopic in D to a point. If $|g(z)| < |f(z)|$ for every z in γ, prove that γ encloses the same number of zeros of f as of $f + g$.

(b) Let g be an analytic function on the disk $|z| < 1 + \epsilon$, for some $\epsilon > 0$. Suppose that g maps the closed unit disk into the open unit disk (both centred at 0). Prove that g has exactly one fixed point in the open unit disk.

(c) Prove that, if $|a| < 1$, then

$$z^m \left(\frac{z - a}{1 - \bar{a}z} \right)^n - a$$

has $m + n$ zeros in $|z| < 1$.

14E Geometry

Show that every isometry of Euclidean space $\mathbb{R}^3$ is a composition of reflections in planes.

What is the smallest integer N such that every isometry f of $\mathbb{R}^3$ with $f(0) = 0$ can be expressed as the composition of at most N reflections? Give an example of an isometry that needs this number of reflections and justify your answer.

Describe (geometrically) all twelve orientation-reversing isometries of a regular tetrahedron.

15H Optimization

Consider the following linear programming problem

$$\begin{array}{ll} \text{maximise} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 - x_2 \geq 1, \\ & 4x_1 - x_2 \geq 10, \\ & x_2 \leq 6, \\ & x_i \geq 0,\ i = 1, 2. \end{array} \tag{1}$$

Write down the Phase One problem for (1) and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve (1), i.e., find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

16B Numerical Analysis

(a) Consider a system of linear equations $Ax = b$ with a non-singular square $n \times n$ matrix A. To determine its solution $x = x^*$ we apply the iterative method

$$x^{k+1} = Hx^k + v.$$

Here $v \in \mathbb{R}^n$, while the matrix $H \in \mathbb{R}^{n \times n}$ is such that $x^* = Hx^* + v$ implies $Ax^* = b$. The initial vector $x^0 \in \mathbb{R}^n$ is arbitrary. Prove that, if the matrix H possesses n linearly independent eigenvectors $w_1, \ldots, w_n$ whose corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy $\max_i |\lambda_i| < 1$, then the method converges for any choice of x^0, i.e. $x^k \to x^*$ as $k \to \infty$.

(b) Describe the Jacobi iteration method for solving $Ax = b$. Show directly from the definition of the method that, if the matrix A is strictly diagonally dominant by rows, i.e.

$$|a_{ii}|^{-1} \sum_{j=1, j \neq i}^{n} |a_{ij}| \leq \gamma < 1, \quad i = 1, \ldots, n,$$

then the method converges.

17F Linear Mathematics

Define the *determinant* of an $n \times n$ matrix A, and prove from your definition that if A' is obtained from A by an elementary row operation (i.e. by adding a scalar multiple of the ith row of A to the jth row, for some $j \neq i$), then $\det A' = \det A$.

Prove also that if X is a $2n \times 2n$ matrix of the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where O denotes the $n \times n$ zero matrix, then $\det X = \det A \det C$. Explain briefly how the $2n \times 2n$ matrix

$$\begin{pmatrix} B & I \\ O & A \end{pmatrix}$$

can be transformed into the matrix

$$\begin{pmatrix} B & I \\ -AB & O \end{pmatrix}$$

by a sequence of elementary row operations. Hence or otherwise prove that $\det AB = \det A \det B$.

18C Fluid Dynamics

Use Euler's equation to derive Bernoulli's theorem for the steady flow of an inviscid fluid of uniform density ρ in the absence of body forces.

Such a fluid flows steadily through a long cylindrical elastic tube having circular cross-section. The variable z measures distance downstream along the axis of the tube. The tube wall has thickness $h(z)$, so that if the external radius of the tube is $r(z)$, its internal radius is $r(z) - h(z)$, where $h(z) \geqslant 0$ is a given slowly-varying function that tends to zero as $z \to \pm\infty$. The elastic tube wall exerts a pressure $p(z)$ on the fluid given as

$$p(z) = p_0 + k[r(z) - R],$$

where p_0, k and R are positive constants. Far upstream, r has the constant value R, the fluid pressure has the constant value p_0, and the fluid velocity u has the constant value V. Assume that gravity is negligible and that $h(z)$ varies sufficiently slowly that the velocity may be taken as uniform across the tube at each value of z. Use mass conservation and Bernoulli's theorem to show that $u(z)$ satisfies

$$\frac{h}{R} = 1 - \left(\frac{V}{u}\right)^{1/2} + \frac{1}{4}\lambda\left[1 - \left(\frac{u}{V}\right)^2\right], \quad \text{where} \quad \lambda = \frac{2\rho V^2}{kR}\ .$$

Sketch a graph of h/R against u/V. Show that if $h(z)$ exceeds a critical value $h_c(\lambda)$, no such flow is possible and find $h_c(\lambda)$.

Show that if $h < h_c(\lambda)$ everywhere, then for given h the equation has two positive solutions for u. Explain how the given value of λ determines which solution should be chosen.

19F Quadratic Mathematics

Explain what is meant by saying that a positive definite integral quadratic form $f(x, y) = ax^2 + bxy + cy^2$ is *reduced*, and show that every positive definite form is equivalent to a reduced form.

State a criterion for a prime number p to be representable by some form of discriminant d, and deduce that p is representable by a form of discriminant -32 if and only if $p \equiv 1, 2$ or $3 \pmod 8$. Find the reduced forms of discriminant -32, and hence or otherwise show that a prime p is representable by the form $3x^2 + 2xy + 3y^2$ if and only if $p \equiv 3 \pmod 8$.

[*Standard results on when -1 and 2 are squares* $\pmod p$ *may be assumed.*]

20D Quantum Mechanics

A quantum mechanical system has two states χ_0 and χ_1, which are normalised energy eigenstates of a Hamiltonian H_3, with

$$H_3\chi_0 = -\chi_0, \quad H_3\chi_1 = +\chi_1.$$

A general time-dependent state may be written

$$\Psi(t) = a_0(t)\chi_0 + a_1(t)\chi_1, \tag{1}$$

where $a_0(t)$ and $a_1(t)$ are complex numbers obeying $|a_0(t)|^2 + |a_1(t)|^2 = 1$.

(a) Write down the time-dependent Schrödinger equation for $\Psi(t)$, and show that if the Hamiltonian is H_3, then

$$i\hbar\frac{da_0}{dt} = -a_0, \qquad i\hbar\frac{da_1}{dt} = +a_1.$$

For the general state given in equation (1) above, write down the probability to observe the system, at time t, in a state $\alpha\chi_0 + \beta\chi_1$, properly normalised so that $|\alpha|^2 + |\beta|^2 = 1$.

(b) Now consider starting the system in the state χ_0 at time $t = 0$, and evolving it with a different Hamiltonian H_1, which acts on the states χ_0 and χ_1 as follows:

$$H_1\chi_0 = \chi_1, \qquad H_1\chi_1 = \chi_0.$$

By solving the time-dependent Schrödinger equation for the Hamiltonian H_1, find $a_0(t)$ and $a_1(t)$ in this case. Hence determine the shortest time $T > 0$ such that $\Psi(T)$ is an eigenstate of H_3 with eigenvalue $+1$.

(c) Now consider taking the state $\Psi(T)$ from part (b), and evolving it for further length of time T, with Hamiltonian H_2, which acts on the states χ_0 and χ_1 as follows:

$$H_2\chi_0 = -i\chi_1, \qquad H_2\chi_1 = i\chi_0.$$

What is the final state of the system? Is this state observationally distinguishable from the original state χ_0?

END OF PAPER