UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 6 June 2002 9 to 12

PAPER 2

Before you begin read these instructions carefully.

*Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Answers must be tied up in separate bundles, marked **A, B, ..., H** according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1E Analysis II

Define what is meant by (i) a complete metric space, and (ii) a totally bounded metric space.

Give an example of a metric space that is complete but not totally bounded. Give an example of a metric space that is totally bounded but not complete.

Give an example of a continuous function that maps a complete metric space onto a metric space that is not complete. Give an example of a continuous function that maps a totally bounded metric space onto a metric space that is not totally bounded.

[*You need not justify your examples.*]

2C Methods

Write down the transformation law for the components of a second-rank tensor A_{ij} explaining the meaning of the symbols that you use.

A tensor is said to have *cubic symmetry* if its components are unchanged by rotations of $\pi/2$ about each of the three co-ordinate axes. Find the most general second-rank tensor having cubic symmetry.

3H Statistics

Explain what is meant by a uniformly most powerful test, its power function and size.

Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables with common density $\rho e^{-\rho y}$, $y \geq 0$. Obtain the uniformly most powerful test of $\rho = \rho_0$ against alternatives $\rho < \rho_0$ and determine the power function of the test.

4G Further Analysis

Let the function $f = u + iv$ be analytic in the complex plane $\mathbb{C}$ with u,v real-valued. Prove that, if uv is bounded above everywhere on $\mathbb{C}$, then f is constant.

5B Numerical Analysis

Applying the Gram–Schmidt orthogonalization, compute a "skinny" QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 6 \\ 1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix},$$

i.e. find a 4×3 matrix Q with orthonormal columns and an upper triangular 3×3 matrix R such that $A = QR$.

6G Linear Mathematics

Let A be a complex 4×4 matrix such that $A^3 = A^2$. What are the possible minimal polynomials of A? If A is not diagonalisable and $A^2 \neq 0$, list all possible Jordan normal forms of A.

7B Complex Methods

Suppose that f is analytic, and that $|f(z)|^2$ is constant in an open disk D. Use the Cauchy–Riemann equations to show that $f(z)$ is constant in D.

8F Quadratic Mathematics

Explain what is meant by a *sesquilinear form* on a complex vector space V. If ϕ and ψ are two such forms, and $\phi(v,v) = \psi(v,v)$ for all $v \in V$, prove that $\phi(v,w) = \psi(v,w)$ for all $v, w \in V$. Deduce that if $\alpha \colon V \to V$ is a linear map satisfying $\phi(\alpha(v), \alpha(v)) = \phi(v,v)$ for all $v \in V$, then $\phi(\alpha(v), \alpha(w)) = \phi(v,w)$ for all $v, w \in V$.

9D Quantum Mechanics

From the expressions

$$L_x = yP_z - zP_y, \quad L_y = zP_x - xP_z, \quad L_z = xP_y - yP_x,$$

show that

$$(x + iy)z$$

is an eigenfunction of $\mathbf{L}^2$ and L_z, and compute the corresponding eigenvalues.

SECTION II

10E Analysis II

(a) Let f be a map of a complete metric space (X,d) into itself, and suppose that there exists some k in $(0,1)$, and some positive integer N, such that $d\big(f^N(x), f^N(y)\big) \leqslant k\,d(x,y)$ for all distinct x and y in X, where f^m is the mth iterate of f. Show that f has a unique fixed point in X.

(b) Let f be a map of a compact metric space (X,d) into itself such that $d\big(f(x), f(y)\big) < d(x,y)$ for all distinct x and y in X. By considering the function $d\big(f(x), x\big)$, or otherwise, show that f has a unique fixed point in X.

(c) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|f(x) - f(y)| < |x - y|$ for every distinct x and y in $\mathbb{R}^n$. Suppose that for some x, the orbit $O(x) = \{x, f(x), f^2(x), \ldots\}$ is bounded. Show that f maps the closure of $O(x)$ into itself, and deduce that f has a unique fixed point in $\mathbb{R}^n$.

[*The Contraction Mapping Theorem may be used without proof providing that it is correctly stated.*]

11C Methods

If $\mathbf{B}$ is a vector, and

$$T_{ij} = \alpha B_i B_j + \beta B_k B_k \delta_{ij} \ ,$$

show for arbitrary scalars α and β that T_{ij} is a symmetric second-rank tensor.

Find the eigenvalues and eigenvectors of T_{ij}.

Suppose now that $\mathbf{B}$ depends upon position $\mathbf{x}$ and that $\nabla \cdot \mathbf{B} = 0$. Find constants α and β such that

$$\frac{\partial}{\partial x_j} T_{ij} = [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i \ .$$

Hence or otherwise show that if $\mathbf{B}$ vanishes everywhere on a surface S that encloses a volume V then

$$\int_V (\nabla \times \mathbf{B}) \times \mathbf{B} \; dV = 0 \ .$$

12H Statistics

For ten steel ingots from a production process the following measures of hardness were obtained:

73.2, 74.3, 75.4, 73.8, 74.4, 76.7, 76.1, 73.0, 74.6, 74.1.

On the assumption that the variation is well described by a normal density function obtain an estimate of the process mean.

The manufacturer claims that he is supplying steel with mean hardness 75. Derive carefully a (generalized) likelihood ratio test of this claim. Knowing that for the data above

$$S_{XX} = \sum_{j=1}^{n} (X_i - \bar{X})^2 = 12.824,$$

what is the result of the test at the 5% significance level?

[

Distribution	t_9	t_{10}
95% percentile	*1.83*	*1.81*
97.5% percentile	*2.26*	*2.23*

]

13G Further Analysis

(a) Given a topology $\mathcal{T}$ on X, a collection $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every non-empty set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$. Prove that a collection $\mathcal{B}$ is a basis for some topology if it satisfies:

(i) the union of all sets in $\mathcal{B}$ is X;

(ii) if $x \in B_1 \cap B_2$ for two sets B_1 and B_2 in $\mathcal{B}$, then there is a set $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

(b) On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consider the dictionary order given by

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. Given points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\mathbf{z} \in \mathbb{R}^2 : \mathbf{x} < \mathbf{z} < \mathbf{y}\}.$$

Show that the sets $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ form a basis of a topology.

(c) Show that this topology on $\mathbb{R}^2$ does not have a countable basis.

14B Numerical Analysis

Let $f \in C[a,b]$ and let $n+1$ distinct points $x_0, \ldots, x_n \in [a,b]$ be given.

(a) Define the divided difference $f[x_0, \ldots, x_n]$ of order n in terms of interpolating polynomials. Prove that it is a symmetric function of the variables x_i, $i = 0, \ldots, n$.

(b) Prove the recurrence relation

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

(c) Hence or otherwise deduce that, for any $i \neq j$, we have

$$f[x_0, \ldots, x_n] = \frac{f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] - f[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]}{x_j - x_i}.$$

(d) From the formulas above, show that, for any $i = 1, \ldots, n-1$,

$$f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] = \gamma f[x_0, \ldots, x_{n-1}] + (1-\gamma) f[x_1, \ldots, x_n],$$

where $\gamma = \dfrac{x_i - x_0}{x_n - x_0}$.

15G Linear Mathematics

(a) A complex $n \times n$ matrix is said to be unipotent if $U - I$ is nilpotent, where I is the identity matrix. Show that U is unipotent if and only if 1 is the only eigenvalue of U.

(b) Let T be an invertible complex matrix. By considering the Jordan normal form of T show that there exists an invertible matrix P such that

$$PTP^{-1} = D_0 + N,$$

where D_0 is an invertible diagonal matrix, N is an upper triangular matrix with zeros in the diagonal and $D_0 N = N D_0$.

(c) Set $D = P^{-1} D_0 P$ and show that $U = D^{-1} T$ is unipotent.

(d) Conclude that any invertible matrix T can be written as $T = DU$ where D is diagonalisable, U is unipotent and $DU = UD$.

16B Complex Methods

A function $f(z)$ has an isolated singularity at a, with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n .$$

(a) Define $\operatorname{res}(f, a)$, the residue of f at the point a.

(b) Prove that if a is a pole of order $k+1$, then

$$\operatorname{res}(f, a) = \lim_{z \to a} \frac{h^{(k)}(z)}{k!}, \quad \text{where} \quad h(z) = (z-a)^{k+1} f(z).$$

(c) Using the residue theorem and the formula above show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{k+1}} = \pi \frac{(2k)!}{(k!)^2} 4^{-k}, \quad k \geq 1 .$$

17F Quadratic Mathematics

Define the *adjoint* α^* of an endomorphism α of a complex inner-product space V. Show that if W is a subspace of V, then $\alpha(W) \subseteq W$ if and only if $\alpha^*(W^{\perp}) \subseteq W^{\perp}$.

An endomorphism of a complex inner-product space is said to be *normal* if it commutes with its adjoint. Prove the following facts about a normal endomorphism α of a finite-dimensional space V.

(i) α and α^* have the same kernel.

(ii) α and α^* have the same eigenvectors, with complex conjugate eigenvalues.

(iii) If $E_\lambda = \{x \in V : \alpha(x) = \lambda x\}$, then $\alpha(E_\lambda^{\perp}) \subseteq E_\lambda^{\perp}$.

(iv) There is an orthonormal basis of V consisting of eigenvectors of α.

Deduce that an endomorphism α is normal if and only if it can be written as a product $\beta\gamma$, where β is Hermitian, γ is unitary and β and γ commute with each other. [*Hint: Given α, define β and γ in terms of their effect on the basis constructed in (iv).*]

18D Quantum Mechanics

Consider a quantum mechanical particle moving in an upside-down harmonic oscillator potential. Its wavefunction $\Psi(x,t)$ evolves according to the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{2}x^2\Psi. \tag{1}$$

(a) Verify that

$$\Psi(x,t) = A(t)e^{-B(t)x^2} \tag{2}$$

is a solution of equation (1), provided that

$$\frac{dA}{dt} = -i\hbar\, AB,$$

and

$$\frac{dB}{dt} = -\frac{i}{2\hbar} - 2i\hbar B^2. \tag{3}$$

(b) Verify that $B = \frac{1}{2\hbar}\tan(\phi - it)$ provides a solution to (3), where ϕ is an arbitrary real constant.

(c) The expectation value of an operator $\mathcal{O}$ at time t is

$$\langle\mathcal{O}\rangle(t) \equiv \int_{-\infty}^{\infty} dx \Psi^*(x,t)\mathcal{O}\Psi(x,t),$$

where $\Psi(x,t)$ is the normalised wave function. Show that for $\Psi(x,t)$ given by (2),

$$\langle x^2\rangle = \frac{1}{4Re(B)}, \quad \langle p^2\rangle = 4\hbar^2|B|^2\langle x^2\rangle.$$

Hence show that as $t\to\infty$,

$$\langle x^2\rangle \approx \langle p^2\rangle \approx \frac{\hbar}{4\sin 2\phi}e^{2t}.$$

[*Hint: You may use*

$$\frac{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}x^2}{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}} = \frac{1}{2C}.]$$

END OF PAPER