UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 3 June 2002 9.00 to 12.00

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I. In Section II at most* ***five*** *answers will be taken into account and no more than* ***three*** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in four bundles, marked ***C***, ***B***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet *must* ***bear your examination number and desk number.***

SECTION I

1C Numbers and Sets

What does it mean to say that a function $f : A \to B$ is injective? What does it mean to say that a function $g : A \to B$ is surjective?

Consider the functions $f : A \to B$, $g : B \to C$ and their composition $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$. Prove the following results.

(i) If f and g are surjective, then so is $g \circ f$.

(ii) If f and g are injective, then so is $g \circ f$.

(iii) If $g \circ f$ is injective, then so is f.

(iv) If $g \circ f$ is surjective, then so is g.

Give an example where $g \circ f$ is injective and surjective but f is not surjective and g is not injective.

2C Numbers and Sets

If f, $g : \mathbb{R} \to \mathbb{R}$ are infinitely differentiable, Leibniz's rule states that, if $n \geqslant 1$,

$$\frac{d^n}{dx^n}\big(f(x)g(x)\big) = \sum_{r=0}^{n} \binom{n}{r} f^{(n-r)}(x)g^{(r)}(x).$$

Prove this result by induction. (You should prove any results on binomial coefficients that you need.)

3E Dynamics

The position x of the leading edge of an avalanche moving down a mountain side making a positive angle α to the horizontal satisfies the equation

$$\frac{d}{dt}\left(x\frac{dx}{dt}\right) = gx\sin\alpha,$$

where g is the acceleration due to gravity.

By multiplying the equation by $x\frac{dx}{dt}$, obtain the first integral

$$x^2\dot{x}^2 = \frac{2g}{3}x^3\sin\alpha + c,$$

where c is an arbitrary constant of integration and the dot denotes differentiation with respect to time.

Sketch the positive quadrant of the $(x,\dot{x})$ phase plane. Show that all solutions approach the trajectory

$$\dot{x} = \left(\frac{2g\sin\alpha}{3}\right)^{\frac{1}{2}} x^{\frac{1}{2}}.$$

Hence show that, independent of initial conditions, the avalanche ultimately has acceleration $\frac{1}{3}g\sin\alpha$.

4E Dynamics

An inertial reference frame S and another reference frame S' have a common origin O. S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to S. Assuming the result that

$$\left(\frac{d\mathbf{a}}{dt}\right)_S = \left(\frac{d\mathbf{a}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}$$

for an arbitrary vector $\mathbf{a}(t)$, show that

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}} = \left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{x}}{dt}\right)_{\mathcal{S}'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}),$$

where $\mathbf{x}$ is the position vector of a point P measured from the origin.

A system of electrically charged particles, all with equal masses m and charges e, moves under the influence of mutual central forces $\mathbf{F}_{ij}$ of the form

$$\mathbf{F}_{ij} = (\mathbf{x}_i - \mathbf{x}_j)f(|\mathbf{x}_i - \mathbf{x}_j|).$$

In addition each particle experiences a Lorentz force due to a constant weak magnetic field $\mathbf{B}$ given by

$$e\frac{d\mathbf{x}_i}{dt} \times \mathbf{B}.$$

Transform the equations of motion to the rotating frame $\mathcal{S}'$. Show that if the angular velocity is chosen to satisfy

$$\boldsymbol{\omega} = -\frac{e}{2m}\mathbf{B},$$

and if terms of second order in $\mathbf{B}$ are neglected, then the equations of motion in the rotating frame are identical to those in the non-rotating frame in the absence of the magnetic field $\mathbf{B}$.

SECTION II

5F Numbers and Sets

What is meant by saying that a set is countable?

Prove that the union of countably many countable sets is itself countable.

Let $\{J_i : i \in I\}$ be a collection of disjoint intervals of the real line, each having strictly positive length. Prove that the index set I is countable.

6F Numbers and Sets

(a) Let S be a finite set, and let $\mathbb{P}(S)$ be the power set of S, that is, the set of all subsets of S. Let $f : \mathbb{P}(S) \to \mathbb{R}$ be additive in the sense that $f(A \cup B) = f(A) + f(B)$ whenever $A \cap B = \varnothing$. Show that, for $A_1, A_2, \ldots, A_n \in \mathbb{P}(S)$,

$$f\left(\bigcup_i A_i\right) = \sum_i f(A_i) - \sum_{i<j} f(A_i \cap A_j) + \sum_{i<j<k} f(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} f\left(\bigcap_i A_i\right).$$

(b) Let $A_1, A_2, \ldots, A_n$ be finite sets. Deduce from part (a) the inclusion–exclusion formula for the size (or cardinality) of $\bigcup_i A_i$.

(c) A *derangement* of the set $S = \{1, 2, \ldots, n\}$ is a permutation π (that is, a bijection from S to itself) in which no member of the set is fixed (that is, $\pi(i) \neq i$ for all i). Using the inclusion–exclusion formula, show that the number d_n of derangements satisfies $d_n/n! \to e^{-1}$ as $n \to \infty$.

7B Numbers and Sets

(a) Suppose that p is an odd prime. Find $1^p + 2^p + \ldots + (p-1)^p$ modulo p.

(b) Find $(p-1)!$ modulo $(1 + 2 + \ldots + (p-1))$, when p is an odd prime.

8B Numbers and Sets

Suppose that a, b are coprime positive integers. Write down an integer $d > 0$ such that $a^d \equiv 1$ modulo b. The least such d is the *order* of a modulo b. Show that if the order of a modulo b is y, and $a^x \equiv 1$ modulo b, then y divides x.

Let $n \geqslant 2$ and $F_n = 2^{2^n} + 1$. Suppose that p is a prime factor of F_n. Find the order of 2 modulo p, and show that $p \equiv 1$ modulo 2^{n+1}.

9E Dynamics

Write down the equations of motion for a system of n gravitating point particles with masses m_i and position vectors $\mathbf{x}_i = \mathbf{x}_i(t)$, $i = 1, 2, \ldots, n$.

Assume that $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Obtain a system of equations for the vectors $\mathbf{a}_i$ which does not involve the time variable t.

Show that the constant vectors $\mathbf{a}_i$ must be located at stationary points of the function

$$\sum_i \frac{1}{9} m_i \, \mathbf{a}_i \cdot \mathbf{a}_i \; + \; \frac{1}{2} \sum_j \sum_{i \neq j} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|} \; .$$

Show that for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other point?

10E Dynamics

Derive the equation

$$\frac{d^2 u}{d\theta^2} + u = \frac{f(u)}{m h^2 u^2} \; ,$$

for the orbit $r^{-1} = u(\theta)$ of a particle of mass m and angular momentum hm moving under a central force $f(u)$ directed towards a fixed point O. Give an interpretation of h in terms of the area swept out by a radius vector.

If the orbits are found to be circles passing through O, then deduce that the force varies inversely as the fifth power of the distance, $f = cu^5$, where c is a constant. Is the force attractive or repulsive?

Show that, for fixed mass, the radius R of the circle varies inversely as the angular momentum of the particle, and hence that the time taken to traverse a complete circle is proportional to R^3.

[*You may assume, if you wish, the expressions for radial and transverse acceleration in the forms* $\ddot{r} - r\dot{\theta}^2, \; 2\dot{r}\dot{\theta} + r\ddot{\theta}$.]

11E Dynamics

An electron of mass m moving with velocity $\dot{\mathbf{x}}$ in the vicinity of the North Pole experiences a force

$$\mathbf{F} = a\dot{\mathbf{x}} \times \frac{\mathbf{x}}{|\mathbf{x}|^3} \ ,$$

where a is a constant and the position vector $\mathbf{x}$ of the particle is with respect to an origin located at the North Pole. Write down the equation of motion of the electron, neglecting gravity. By taking the dot product of the equation with $\dot{\mathbf{x}}$ show that the speed of the electron is constant. By taking the cross product of the equation with $\mathbf{x}$ show that

$$m\mathbf{x} \times \dot{\mathbf{x}} - a\frac{\mathbf{x}}{|\mathbf{x}|} = \mathbf{L} \ ,$$

where $\mathbf{L}$ is a constant vector. By taking the dot product of this equation with $\mathbf{x}$, show that the electron moves on a cone centred on the North Pole.

12E Dynamics

Calculate the moment of inertia of a uniform rod of length $2l$ and mass M about an axis through its centre and perpendicular to its length. Assuming it moves in a plane, give an expression for the kinetic energy of the rod in terms of the speed of the centre and the angle that it makes with a fixed direction.

Two such rods are freely hinged together at one end and the other two ends slide on a perfectly smooth horizontal floor. The rods are initially at rest and lie in a vertical plane, each making an angle α to the horizontal. The rods subsequently move under gravity. Calculate the speed with which the hinge strikes the ground.

END OF PAPER