UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 30 May 2002 9.00 to 12.00

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I. In Section II at most* ***five*** *answers will be taken into account and no more than* ***three*** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in three bundles, marked ***B***, ***C*** *and* ***D*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet *must* ***bear your examination number and desk number.***

SECTION I

1B Algebra and Geometry

(a) State the Orbit-Stabilizer Theorem for a finite group G acting on a set X.

(b) Suppose that G is the group of rotational symmetries of a cube C. Two regular tetrahedra T and T' are inscribed in C, each using half the vertices of C. What is the order of the stabilizer in G of T?

2D Algebra and Geometry

State the Fundamental Theorem of Algebra. Define the characteristic equation for an arbitrary 3×3 matrix A whose entries are complex numbers. Explain why the matrix must have three eigenvalues, not necessarily distinct.

Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

and hence find the three eigenvalues of A. Find a set of linearly independent eigenvectors, specifying which eigenvector belongs to which eigenvalue.

3C Analysis I

Suppose $a_n \in \mathbb{R}$ for $n \geqslant 1$ and $a \in \mathbb{R}$. What does it mean to say that $a_n \to a$ as $n \to \infty$? What does it mean to say that $a_n \to \infty$ as $n \to \infty$?

Show that, if $a_n \neq 0$ for all n and $a_n \to \infty$ as $n \to \infty$, then $1/a_n \to 0$ as $n \to \infty$. Is the converse true? Give a proof or a counter example.

Show that, if $a_n \neq 0$ for all n and $a_n \to a$ with $a \neq 0$, then $1/a_n \to 1/a$ as $n \to \infty$.

4C Analysis I

Show that any bounded sequence of real numbers has a convergent subsequence.

Give an example of a sequence of real numbers with no convergent subsequence.

Give an example of an unbounded sequence of real numbers with a convergent subsequence.

SECTION II

5B Algebra and Geometry

(a) Find a subset T of the Euclidean plane $\mathbb{R}^2$ that is not fixed by any isometry (rigid motion) except the identity.

Let G be a subgroup of the group of isometries of $\mathbb{R}^2$, T a subset of $\mathbb{R}^2$ not fixed by any isometry except the identity, and let S denote the union $\bigcup_{g \in G} g(T)$. Does the group H of isometries of S contain G? Justify your answer.

(b) Find an example of such a G and T with $H \neq G$.

6B Algebra and Geometry

(a) Suppose that g is a Möbius transformation, acting on the extended complex plane. What are the possible numbers of fixed points that g can have? Justify your answer.

(b) Show that the operation c of complex conjugation, defined by $c(z) = \bar{z}$, is not a Möbius transformation.

7B Algebra and Geometry

(a) Find, with justification, the matrix, with respect to the standard basis of $\mathbb{R}^2$, of the rotation through an angle α about the origin.

(b) Find the matrix, with respect to the standard basis of $\mathbb{R}^3$, of the rotation through an angle α about the axis containing the point $(\frac{3}{5}, \frac{4}{5}, 0)$ and the origin. You may express your answer in the form of a product of matrices.

8D Algebra and Geometry

Define what is meant by a vector space V over the real numbers $\mathbb{R}$. Define subspace, proper subspace, spanning set, basis, and dimension.

Define the sum $U + W$ and intersection $U \cap W$ of two subspaces U and W of a vector space V. Why is the intersection never empty?

Let $V = \mathbb{R}^4$ and let $U = \{\mathbf{x} \in V : x_1 - x_2 + x_3 - x_4 = 0\}$, where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, and let $W = \{\mathbf{x} \in V : x_1 - x_2 - x_3 + x_4 = 0\}$. Show that $U \cap W$ has the orthogonal basis $\mathbf{b}_1$, $\mathbf{b}_2$ where $\mathbf{b}_1 = (1, 1, 0, 0)$ and $\mathbf{b}_2 = (0, 0, 1, 1)$. Extend this basis to find orthogonal bases of U, W, and $U + W$. Show that $U + W = V$ and hence verify that, in this case,

$$\dim U + \dim W \;=\; \dim(U + W) + \dim(U \cap W)\ .$$

9C Analysis I

State some version of the fundamental axiom of analysis. State the alternating series test and prove it from the fundamental axiom.

In each of the following cases state whether $\sum_{n=1}^{\infty} a_n$ converges or diverges and prove your result. You may use any test for convergence provided you state it correctly.

(i) $a_n = (-1)^n (\log(n+1))^{-1}$.

(ii) $a_{2n} = (2n)^{-2}$, $a_{2n-1} = -n^{-2}$.

(iii) $a_{3n-2} = -(2n-1)^{-1}$, $a_{3n-1} = (4n-1)^{-1}$, $a_{3n} = (4n)^{-1}$.

(iv) $a_{2^n+r} = (-1)^n (2^n + r)^{-1}$ for $0 \leqslant r \leqslant 2^n - 1$, $n \geqslant 0$.

10C Analysis I

Show that a continuous real-valued function on a closed bounded interval is bounded and attains its bounds.

Write down examples of the following functions (no proof is required).

(i) A continuous function $f_1 : (0,1) \to \mathbb{R}$ which is not bounded.

(ii) A continuous function $f_2 : (0,1) \to \mathbb{R}$ which is bounded but does not attain its bounds.

(iii) A bounded function $f_3 : [0,1] \to \mathbb{R}$ which is not continuous.

(iv) A function $f_4 : [0,1] \to \mathbb{R}$ which is not bounded on any interval $[a,b]$ with $0 \leqslant a < b \leqslant 1$.

[*Hint: Consider first how to define f_4 on the rationals.*]

11C Analysis I

State the mean value theorem and deduce it from Rolle's theorem.

Use the mean value theorem to show that, if $h : \mathbb{R} \to \mathbb{R}$ is differentiable with $h'(x) = 0$ for all x, then h is constant.

By considering the derivative of the function g given by $g(x) = e^{-ax} f(x)$, find all the solutions of the differential equation $f'(x) = af(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is differentiable and a is a fixed real number.

Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \int_0^x f(t)\, dt$$

is differentiable with $F'(x) = f(x)$.

Find the solution of the equation

$$g(x) = A + \int_0^x g(t)\, dt$$

where $g : \mathbb{R} \to \mathbb{R}$ is differentiable and A is a real number. You should explain why the solution is unique.

12C Analysis I

Prove Taylor's theorem with some form of remainder.

An infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation

$$f^{(3)}(x) = f(x)$$

and the conditions $f(0) = 1$, $f'(0) = f''(0) = 0$. If $R > 0$ and j is a positive integer, explain why we can find an M_j such that

$$|f^{(j)}(x)| \leqslant M_j$$

for all x with $|x| \leqslant R$. Explain why we can find an M such that

$$|f^{(j)}(x)| \leqslant M$$

for all x with $|x| \leqslant R$ and all $j \geqslant 0$.

Use your form of Taylor's theorem to show that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \, .$$

END OF PAPER