UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

List of Courses

Algebra and Geometry
Analysis I
Differential Equations
Dynamics
Numbers and Sets
Probability
Vector Calculus

1/I/1C **Algebra and Geometry**

Show, using the summation convention or otherwise, that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$, for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

The function $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\Pi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n})$ where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. Show that Π is linear and find the elements of a matrix P such that $\Pi(\mathbf{x}) = P\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Find all solutions to the equation $\Pi(\mathbf{x}) = \mathbf{x}$. Evaluate $\Pi(\mathbf{n})$. Describe the function Π geometrically. Justify your answer.

1/I/2C **Algebra and Geometry**

Define what is meant by the statement that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ are linearly independent. Determine whether the following vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ are linearly independent and justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

For the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ taken from a real vector space V consider the statements

A) $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent,
B) $\exists\, \alpha, \beta, \gamma \in \mathbb{R} : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
C) $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$, not all $= 0 : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
D) $\exists\, \alpha, \beta \in \mathbb{R}$, not both $= 0 : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
E) $\exists\, \alpha, \beta \in \mathbb{R} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
F) $\nexists$ basis of V that contains all 3 vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

State if the following implications are true or false (no justification is required):

i)	A $\Rightarrow$ B,	vi)	B $\Rightarrow$ A,
ii)	A $\Rightarrow$ C,	vii)	C $\Rightarrow$ A,
iii)	A $\Rightarrow$ D,	viii)	D $\Rightarrow$ A,
iv)	A $\Rightarrow$ E,	ix)	E $\Rightarrow$ A,
v)	A $\Rightarrow$ F,	x)	F $\Rightarrow$ A.

1/II/5C **Algebra and Geometry**

The matrix

$$A_\alpha = \begin{pmatrix} 1 & -1 & 2\alpha+1 \\ 1 & \alpha-1 & 1 \\ 1+\alpha & -1 & \alpha^2+4\alpha+1 \end{pmatrix}$$

defines a linear map $\Phi_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_\alpha(\mathbf{x}) = A_\alpha \mathbf{x}$. Find a basis for the kernel of Φ_α for all values of $\alpha \in \mathbb{R}$.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be bases of $\mathbb{R}^3$. Show that there exists a matrix S, to be determined in terms of $\mathcal{B}$ and $\mathcal{C}$, such that, for every linear mapping Φ, if Φ has matrix A with respect to $\mathcal{B}$ and matrix A' with respect to $\mathcal{C}$, then $A' = S^{-1}AS$.

For the bases

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad , \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\} \quad ,$$

find the basis transformation matrix S and calculate $S^{-1}A_0S$.

1/II/6C **Algebra and Geometry**

Assume that $\mathbf{x}_p$ is a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x}, \mathbf{b} \in \mathbb{R}^3$ and a real 3×3 matrix A. Explain why the general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = \mathbf{x}_p + \mathbf{h}$ where $\mathbf{h}$ is any vector such that $A\mathbf{h} = \mathbf{0}$.

Now assume that A is a real symmetric 3×3 matrix with three different eigenvalues λ_1, λ_2 and λ_3. Show that eigenvectors of A with respect to different eigenvalues are orthogonal. Let $\mathbf{x}_k$ be a normalised eigenvector of A with respect to the eigenvalue λ_k, $k = 1, 2, 3$. Show that the linear system

$$(A - \lambda_k I)\mathbf{x} \;=\; \mathbf{b} \;\; ,$$

where I denotes the 3×3 unit matrix, is solvable if and only if $\mathbf{x}_k.\mathbf{b} = 0$. Show that the general solution is given by

$$\mathbf{x} \;=\; \sum_{i \neq k} \tfrac{\mathbf{b}.\mathbf{x}_i}{\lambda_i - \lambda_k} \mathbf{x}_i + \beta \mathbf{x}_k, \quad \beta \in \mathbb{R}.$$

[*Hint: consider the components of* $\mathbf{x}$ *and* $\mathbf{b}$ *with respect to a basis of eigenvectors of* A.]

Consider the matrix A and the vector $\mathbf{b}$

$$A = \begin{pmatrix} -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \sqrt{2} + \sqrt{3} \\ -\sqrt{2} + \sqrt{3} \\ -2\sqrt{3} \end{pmatrix}.$$

Verify that $\frac{1}{\sqrt{3}}(1, 1, 1)^T$ and $\frac{1}{\sqrt{2}}(1, -1, 0)^T$ are eigenvectors of A. Show that $A\mathbf{x} = \mathbf{b}$ is solvable and find its general solution.

1/II/7C **Algebra and Geometry**

For $\alpha, \gamma \in \mathbb{R}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$ and $\beta\overline{\beta} \geqslant \alpha\gamma$ the equation $\alpha z\overline{z} - \beta\overline{z} - \overline{\beta}z + \gamma = 0$ describes a circle $C_{\alpha\beta\gamma}$ in the complex plane. Find its centre and radius. What does the equation describe if $\beta\overline{\beta} < \alpha\gamma$? Sketch the circles $C_{\alpha\beta\gamma}$ for $\beta = \gamma = 1$ and $\alpha = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1$.

Show that the complex function $f(z) = \beta\overline{z}/\overline{\beta}$ for $\beta \neq 0$ satisfies $f(C_{\alpha\beta\gamma}) = C_{\alpha\beta\gamma}$.

[*Hint:* $f(C) = C$ *means that* $f(z) \in C\ \forall z \in C$ *and* $\forall w \in C\ \ \exists z \in C$ *such that* $f(z) = w$.]

For two circles C_1 and C_2 a function $m(C_1, C_2)$ is defined by

$$m(C_1, C_2) \ = \ \max_{z \in C_1, w \in C_2} \left| z - w \right| \ .$$

Prove that $m(C_1, C_2) \leqslant m(C_1, C_3) + m(C_2, C_3)$. Show that

$$m(C_{\alpha_1\beta_1\gamma_1}, C_{\alpha_2\beta_2\gamma_2}) \ = \ \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{|\alpha_1\alpha_2|} + \frac{\sqrt{\beta_1\overline{\beta_1} - \alpha_1\gamma_1}}{|\alpha_1|} + \frac{\sqrt{\beta_2\overline{\beta_2} - \alpha_2\gamma_2}}{|\alpha_2|} \ .$$

1/II/8C **Algebra and Geometry**

Let $l_{\mathbf{x}}$ denote the straight line through $\mathbf{x}$ with directional vector $\mathbf{u} \neq \mathbf{0}$

$$l_{\mathbf{x}} \ = \ \left\{ \mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = \mathbf{x} + \lambda\mathbf{u}, \lambda \in \mathbb{R} \right\} \ .$$

Show that $l_{\mathbf{0}}$ is a subspace of $\mathbb{R}^3$ and show that $l_{\mathbf{x}_1} = l_{\mathbf{x}_2} \Leftrightarrow \mathbf{x_1} = \mathbf{x}_2 + \lambda\mathbf{u}$ for some $\lambda \in \mathbb{R}$.

For fixed $\mathbf{u} \neq \mathbf{0}$ let $\mathcal{L}$ be the set of all the parallel straight lines $l_{\mathbf{x}}$ ($\mathbf{x} \in \mathbb{R}^3$) with directional vector $\mathbf{u}$. On $\mathcal{L}$ an addition and a scalar multiplication are defined by

$$l_{\mathbf{x}} + l_{\mathbf{y}} = l_{\mathbf{x}+\mathbf{y}}, \ \alpha l_{\mathbf{x}} = l_{\alpha\mathbf{x}}, \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \ \alpha \in \mathbb{R} \ .$$

Explain why these operations are well-defined. Show that the addition is associative and that there exists a zero vector which should be identified.

You may now assume that $\mathcal{L}$ is a vector space. If $\{\mathbf{u}, \mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^3$ show that $\{l_{\mathbf{b}_1}, l_{\mathbf{b}_2}\}$ is a basis for $\mathcal{L}$.

For $\mathbf{u} = (1, 3, -1)^T$ a linear map $\Phi : \mathcal{L} \to \mathcal{L}$ is defined by

$$\Phi\left(l_{(1,-1,0)^T}\right) = l_{(2,4,-1)^T} \ , \ \Phi\left(l_{(1,1,0)^T}\right) = l_{(-4,-2,1)^T} \ .$$

Find the matrix A of Φ with respect to the basis $\{l_{(1,0,0)^T}, l_{(0,1,0)^T}\}$.

3/I/1F **Algebra and Geometry**

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 = 0$ if and only if $a = -d$ and $bc = -a^2$. Prove that $A^3 = 0$ if and only if $A^2 = 0$.

[*Hint: it is easy to check that* $A^2 - (a+d)A + (ad - bc)I = 0$.]

3/I/2D **Algebra and Geometry**

Show that the set of Möbius transformations of the extended complex plane $\mathbb{C} \cup \{\infty\}$ form a group. Show further that an arbitrary Möbius transformation can be expressed as the composition of maps of the form

$$f(z) = z + a, \quad g(z) = kz \text{ and } h(z) = 1/z\ .$$

3/II/5F **Algebra and Geometry**

Let A, B, C be 2×2 matrices, real or complex. Define the trace $\operatorname{tr} C$ to be the sum of diagonal entries $C_{11} + C_{22}$. Define the commutator $[A, B]$ to be the difference $AB - BA$. Give the definition of the eigenvalues of a 2×2 matrix and prove that it can have at most two distinct eigenvalues. Prove that

a) $\operatorname{tr}[A, B] = 0$,

b) $\operatorname{tr} C$ equals the sum of the eigenvalues of C,

c) if all eigenvalues of C are equal to 0 then $C^2 = 0$,

d) either $[A, B]$ is a diagonalisable matrix or the square $[A, B]^2 = 0$,

e) $[A, B]^2 = \alpha I$ where $\alpha \in \mathbb{C}$ and I is the unit matrix.

3/II/6E **Algebra and Geometry**

Define the notion of an *action* of a group G on a set X. Define *orbit* and *stabilizer*, and then, assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Show that the group of rotations of a cube has order 24.

3/II/7E **Algebra and Geometry**

State Lagrange's theorem. Use it to describe all groups of order p, where p is a fixed prime number.

Find all the subgroups of a fixed cyclic group $\langle x \rangle$ of order n.

3/II/8D **Algebra and Geometry**

(i) Let A_4 denote the alternating group of even permutations of four symbols. Let X be the 3-cycle (123) and P, Q be the pairs of transpositions (12)(34) and (13)(24). Find $X^3, P^2, Q^2, X^{-1}PX, X^{-1}QX$, and show that A_4 is generated by X, P and Q.

(ii) Let G and H be groups and let

$$G \times H = \{(g, h) : g \in G, h \in H\}.$$

Show how to make $G \times H$ into a group in such a way that $G \times H$ contains subgroups isomorphic to G and H.

If D_n is the dihedral group of order n and C_2 is the cyclic group of order 2, show that D_{12} is isomorphic to $D_6 \times C_2$. Is the group D_{12} isomorphic to A_4?

1/I/3D Analysis I

What does it mean to say that $u_n \to l$ as $n \to \infty$?

Show that, if $u_n \to l$ and $v_n \to k$, then $u_n v_n \to lk$ as $n \to \infty$.

If further $u_n \neq 0$ for all n and $l \neq 0$, show that $1/u_n \to 1/l$ as $n \to \infty$.

Give an example to show that the non-vanishing of u_n for all n need not imply the non-vanishing of l.

1/I/4D Analysis I

Starting from the theorem that any continuous function on a closed and bounded interval attains a maximum value, prove Rolle's Theorem. Deduce the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(t) > 0$ for all t show that f is a strictly increasing function.

Conversely, if f is strictly increasing, is $f'(t) > 0$ for all t?

1/II/9D Analysis I

(i) If $a_0, a_1, \ldots$ are complex numbers show that if, for some $w \in \mathbb{C}, w \neq 0$, the set $\{|a_n w^n| : n \geq 0\}$ is bounded and $|z| < |w|$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Use this result to define the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$.

(ii) If $|a_n|^{1/n} \to R$ as $n \to \infty$ $(0 < R < \infty)$ show that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence equal to $1/R$.

(iii) Give examples of power series with radii of convergence 1 such that (a) the series converges at all points of the circle of convergence, (b) diverges at all points of the circle of convergence, and (c) neither of these occurs.

1/II/10D Analysis I

Suppose that f is a continuous real-valued function on $[a, b]$ with $f(a) < f(b)$. If $f(a) < v < f(b)$ show that there exists c with $a < c < b$ and $f(c) = v$.

Deduce that if f is a continuous function from the closed bounded interval $[a, b]$ to itself, there exists at least one fixed point, i.e., a number d belonging to $[a, b]$ with $f(d) = d$. Does this fixed point property remain true if f is a continuous function defined (i) on the open interval (a, b) and (ii) on $\mathbb{R}$? Justify your answers.

1/II/11D **Analysis I**

(i) Show that if $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable then, given $\epsilon > 0$, we can find some constant L and $\delta(\epsilon) > 0$ such that

$$|g(t) - g(\alpha) - g'(\alpha)\,(t-\alpha)| \;\leq\; L|t-\alpha|^2$$

for all $|t - \alpha| < \delta(\epsilon)$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable on $[a, b]$ (with one-sided derivatives at the end points), let f' and f'' be strictly positive functions and let $f(a) < 0 < f(b)$.

If $F(t) = t - (f(t)/f'(t))$ and a sequence $\{x_n\}$ is defined by $b = x_0, x_n = F(x_{n-1})$ $(n > 0)$, show that $x_0, x_1, x_2, \ldots$ is a decreasing sequence of points in $[a, b]$ and hence has limit α. What is $f(\alpha)$? Using part (i) or otherwise estimate the rate of convergence of x_n to α, i.e., the behaviour of the absolute value of $(x_n - \alpha)$ for large values of n.

1/II/12D **Analysis I**

Explain what it means for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$, and give an example of a bounded function that is not Riemann integrable.

Show each of the following statements is true for continuous functions f, but false for general Riemann integrable functions f.

(i) If $f : [a, b] \to \mathbb{R}$ is such that $f(t) \geq 0$ for all t in $[a, b]$ and $\int_a^b f(t)\,dt = 0$, then $f(t) = 0$ for all t in $[a, b]$.

(ii) $\int_a^t f(x)\,dx$ is differentiable and $\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$.

2/I/1B **Differential Equations**

Find the solution to

$$\frac{dy(x)}{dx} + \tanh(x)\, y(x) \;=\; H(x)\;,$$

in the range $-\infty < x < \infty$ subject to $y(0) = 1$, where $H(x)$ is the Heavyside function defined by

$$H(x) \;=\; \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}\;.$$

Sketch the solution.

2/I/2B **Differential Equations**

The function $y(x)$ satisfies the inhomogeneous second-order linear differential equation

$$y'' - y' - 2y \;=\; 18xe^{-x}\;.$$

Find the solution that satisfies the conditions that $y(0) = 1$ and $y(x)$ is bounded as $x \to \infty$.

2/II/5B **Differential Equations**

The real sequence $y_k,\ k = 1, 2, \ldots$ satisfies the difference equation

$$y_{k+2} - y_{k+1} + y_k \ = \ 0 \ .$$

Show that the general solution can be written

$$y_k \ = \ a \cos \frac{\pi k}{3} \ + \ b \sin \frac{\pi k}{3} \ ,$$

where a and b are arbitrary real constants.

Now let y_k satisfy

$$y_{k+2} - y_{k+1} + y_k \ = \ \frac{1}{k+2} \ . \qquad (*)$$

Show that a particular solution of $(*)$ can be written in the form

$$y_k \ = \ \sum_{n=1}^{k} \frac{a_n}{k-n+1} \ ,$$

where

$$a_{n+2} - a_{n+1} + a_n = 0 \ , \qquad n \geq 1 \ ,$$

and $a_1 = 1,\ a_2 = 1$.

Hence, find the general solution to $(*)$.

2/II/6B **Differential Equations**

The function $y(x)$ satisfies the linear equation

$$y''(x) + p(x)y'(x) + q(x)y(x) \;=\; 0\,.$$

The Wronskian, $W(x)$, of two independent solutions denoted $y_1(x)$ and $y_2(x)$ is defined to be

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}\,.$$

Let $y_1(x)$ be given. In this case, show that the expression for $W(x)$ can be interpreted as a first-order inhomogeneous differential equation for $y_2(x)$. Hence, by explicit derivation, show that $y_2(x)$ may be expressed as

$$y_2(x) \;=\; y_1(x)\int_{x_0}^{x}\frac{W(t)}{y_1(t)^2}\,dt\,, \qquad (*)$$

where the rôle of x_0 should be briefly elucidated.

Show that $W(x)$ satisfies

$$\frac{dW(x)}{dx} + p(x)W(x) \;=\; 0\,.$$

Verify that $y_1(x) = 1 - x$ is a solution of

$$xy''(x) - (1-x^2)y'(x) - (1+x)y(x) \;=\; 0\,. \qquad (\dagger)$$

Hence, using $(*)$ with $x_0 = 0$ and expanding the integrand in powers of t to order t^3, find the first three non-zero terms in the power series expansion for a solution, $y_2(x)$, of $(\dagger)$ that is independent of $y_1(x)$ and satisfies $y_2(0) = 0,\ y_2{}''(0) = 1$.

2/II/7B **Differential Equations**

Consider the linear system

$$\dot{\mathbf{z}} + A\mathbf{z} \;=\; \mathbf{h}\,, \qquad (*)$$

where

$$\mathbf{z}(t) \;=\; \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\,, \quad A \;=\; \begin{pmatrix} 1+a & -2 \\ 1 & -1+a \end{pmatrix}\,, \quad \mathbf{h}(t) \;=\; \begin{pmatrix} 2\cos t \\ \cos t - \sin t \end{pmatrix}\,,$$

where $\mathbf{z}(t)$ is real and a is a real constant, $a \geq 0$.

Find a (complex) eigenvector, $\mathbf{e}$, of A and its corresponding (complex) eigenvalue, l. Show that the second eigenvector and corresponding eigenvalue are respectively $\bar{\mathbf{e}}$ and $\bar{l}$, where the bar over the symbols signifies complex conjugation. Hence explain how the general solution to $(*)$ can be written as

$$\mathbf{z}(t) \;=\; \alpha(t)\,\mathbf{e} \;+\; \bar{\alpha}(t)\,\bar{\mathbf{e}}\,,$$

where $\alpha(t)$ is complex.

Write down a differential equation for $\alpha(t)$ and hence, for $a > 0$, deduce the solution to $(*)$ which satisfies the initial condition $\mathbf{z}(0) = \underline{0}$.

Is the linear system resonant?

By taking the limit $a \to 0$ of the solution already found deduce the solution satisfying $\mathbf{z}(0) = \underline{0}$ when $a = 0$.

2/II/8B **Differential Equations**

Carnivorous hunters of population h prey on vegetarians of population p. In the absence of hunters the prey will increase in number until their population is limited by the availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$\begin{aligned} \dot{p} &= p\left(1 - \frac{p}{a}\right) - \frac{ph}{a}\,, \\ \dot{h} &= \frac{h}{8}\left(\frac{p}{b} - 1\right)\,, \end{aligned} \qquad (*)$$

where a and b are positive constants, and $h(t)$ and $p(t)$ are non-negative functions of time, t. By giving an interpretation of each term explain briefly how these equations model the system described.

Consider these equations for $a = 1$. In the two cases $0 < b < 1/2$ and $b > 1$ determine the location and the stability properties of the critical points of $(*)$. In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

4/I/3A **Dynamics**

Derive the equation

$$\frac{d^2u}{d\theta^2}+u=\frac{f(u)}{mh^2u^2}$$

for the motion of a particle of mass m under an attractive central force f, where $u=1/r$ and r is the distance of the particle from the centre of force, and where mh is the angular momentum of the particle about the centre of force.

[*Hint: you may assume the expressions for the radial and transverse accelerations in the form* $\ddot{r}-r\dot{\theta}^2, 2\dot{r}\dot{\theta}+r\ddot{\theta}$.]

4/I/4A **Dynamics**

Two particles of masses m_1 and m_2 at positions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are subject to forces $\mathbf{F}_1=-\mathbf{F}_2=\mathbf{f}(\mathbf{x}_1-\mathbf{x}_2)$. Show that the centre of mass moves at a constant velocity. Obtain the equation of motion for the relative position of the particles. How does the reduced mass

$$\mu=\frac{m_1m_2}{m_1+m_2}$$

of the system enter?

4/II/9A **Dynamics**

The position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. Write down the equation of motion of the particle under a force $\mathbf{F}=-4m\omega^2\mathbf{x}$.

Find the motion of the particle in (x,y,z) coordinates with initial condition

$$\mathbf{x}=(1,0,0) \quad \text{and} \quad \dot{\mathbf{x}}=(0,0,0) \qquad \text{at } t=0,$$

where $\boldsymbol{\omega}=(0,0,\omega)$. Show that the particle has a maximum speed at $t=(2n+1)\pi/4\omega$, and find this speed.

[*Hint: you may find it useful to consider the combination* $\zeta=x+iy$.]

4/II/10A **Dynamics**

A spherical raindrop of radius $a(t) > 0$ and density ρ falls down at a velocity $v(t) > 0$ through a fine stationary mist. As the raindrop falls its volume grows at the rate $c\pi a^2 v$ with constant c. The raindrop is subject to the gravitational force and a resistive force $-k\rho\pi a^2 v^2$ with k a positive constant. Show a and v satisfy

$$\begin{aligned}\dot{a} &= \tfrac{1}{4}cv,\\ \dot{v} &= g - \tfrac{3}{4}(c+k)\frac{v^2}{a}.\end{aligned}$$

Find an expression for $\frac{d}{dt}(v^2/a)$, and deduce that as time increases v^2/a tends to the constant value $g/(\frac{7}{8}c + \frac{3}{4}k)$, and thence the raindrop tends to a constant acceleration which is less than $\frac{1}{7}g$.

4/II/11A **Dynamics**

A spacecraft of mass m moves under the gravitational influence of the Sun of mass M and with universal gravitation constant G. After a disastrous manoeuvre, the unfortunate spacecraft finds itself exactly in a parabolic orbit about the Sun: the orbit with zero total energy. Using the conservation of energy and angular momentum, or otherwise, show that in the subsequent motion the distance of the spacecraft from the Sun $r(t)$ satisfies

$$(r - r_0)(r + 2r_0)^2 = \tfrac{9}{2}GM(t - t_0)^2,$$

with constants r_0 and t_0.

4/II/12A **Dynamics**

Find the moment of inertia of a uniform solid cylinder of radius a, length l and total mass M about its axis.

The cylinder is released from rest at the top of an inclined plane of length L and inclination θ to the horizontal. The first time the plane is perfectly smooth and the cylinder slips down the plane without rotating. The experiment is then repeated after the plane has been roughened, so that the cylinder now rolls without slipping at the point of contact. Show that the time taken to roll down the roughened plane is $\sqrt{\frac{3}{2}}$ times the time taken to slip down the smooth plane.

4/I/1E **Numbers and Sets**

(a) Show that, given a set X, there is no bijection between X and its power set.

(b) Does there exist a set whose members are precisely those sets that are not members of themselves? Justify your answer.

4/I/2E **Numbers and Sets**

Prove, by induction or otherwise, that

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+m}{m} = \binom{n+m+1}{m}.$$

Find the number of sequences consisting of zeroes and ones that contain exactly n zeroes and at most m ones.

4/II/5E **Numbers and Sets**

(a) Prove Wilson's theorem, that $(p-1)! \equiv -1 \pmod{p}$, where p is prime.

(b) Suppose that p is an odd prime. Express $1^2.3^2.5^2.\ldots.(p-2)^2 \pmod{p}$ as a power of -1.

[*Hint:* $k \equiv -(p-k) \pmod{p}$.]

4/II/6E **Numbers and Sets**

State and prove the principle of inclusion-exclusion. Use it to calculate $\phi(4199)$, where ϕ is Euler's ϕ-function.

In a certain large college, a survey revealed that 90% of the fellows detest at least one of the pop stars Hairy, Dirty and Screamer. 45% detest Hairy, 28% detest Dirty and 46% detest Screamer. If 27% detest only Screamer and 6% detest all three, what proportion detest Hairy and Dirty but not Screamer?

4/II/7E **Numbers and Sets**

(a) Prove that, if p is prime and a is not a multiple of p, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) The *order* of $a \pmod{p}$ is the least positive integer d such that $a^d \equiv 1 \pmod{p}$. Suppose now that $a^x \equiv 1 \pmod{p}$; what can you say about x in terms of d? Show that $p \equiv 1 \pmod{d}$.

(c) Suppose that p is an odd prime. What is the order of $x \pmod{p}$ if $x^2 \equiv -1 \pmod{p}$? Find a condition on $p \pmod{4}$ that is equivalent to the existence of an integer x with $x^2 \equiv -1 \pmod{p}$.

4/II/8E **Numbers and Sets**

What is the Principle of Mathematical Induction? Derive it from the statement that every non-empty set of positive integers has a least element.

Prove, by induction on n, that $9^n \equiv 2^n \pmod{7}$ for all $n \geq 1$.

What is wrong with the following argument?

"Theorem: $\sum_{i=1}^{n} i = n(n+1)/2 + 126$.

Proof: Assume that $m \geq 1$ and $\sum_{i=1}^{m} i = m(m+1)/2 + 126$. Add $m+1$ to both sides to get

$$\sum_{i=1}^{m+1} i \;=\; m(m+1)/2 + m + 1 + 126 \;=\; (m+1)(m+2)/2 + 126.$$

So, by induction, the theorem is proved."

2/I/3F **Probability**

The following problem is known as Bertrand's paradox. A chord has been chosen at random in a circle of radius r. Find the probability that it is longer than the side of the equilateral triangle inscribed in the circle. Consider three different cases:

a) the middle point of the chord is distributed uniformly inside the circle,

b) the two endpoints of the chord are independent and uniformly distributed over the circumference,

c) the distance between the middle point of the chord and the centre of the circle is uniformly distributed over the interval $[0, r]$.

[*Hint: drawing diagrams may help considerably.*]

2/I/4F **Probability**

The Ruritanian authorities decided to pardon and release one out of three remaining inmates, A, B and C, kept in strict isolation in the notorious Alkazaf prison. The inmates know this, but can't guess who among them is the lucky one; the waiting is agonising. A sympathetic, but corrupted, prison guard approaches A and offers to name, in exchange for a fee, another inmate (not A) who is doomed to stay. He says: "This reduces your chances to remain here from 2/3 to 1/2: will it make you feel better?" A hesitates but then accepts the offer; the guard names B.

Assume that indeed B will not be released. Determine the conditional probability

$$P\big(A \text{ remains } \big| \ B \text{ named}\big) = \frac{P(A\&B \text{ remain})}{P(B \text{ named})}$$

and thus check the guard's claim, in three cases:

a) when the guard is completely unbiased (i.e., names any of B and C with probability 1/2 if the pair B, C is to remain jailed),

b) if he hates B and would certainly name him if B is to remain jailed,

c) if he hates C and would certainly name him if C is to remain jailed.

2/II/9F **Probability**

I play tennis with my parents; the chances for me to win a game against Mum (M) are p and against Dad (D) q, where $0 < q < p < 1$. We agreed to have three games, and their order can be DMD (where I play against Dad, then Mum then again Dad) or MDM. The results of games are independent.

Calculate under each of the two orders the probabilities of the following events:

a) that I win at least one game,

b) that I win at least two games,

c) that I win at least two games in succession (i.e., games 1 and 2 or 2 and 3, or 1, 2 and 3),

d) that I win exactly two games in succession (i.e., games 1 and 2 or 2 and 3, but not 1, 2 and 3),

e) that I win exactly two games (i.e., 1 and 2 or 2 and 3 or 1 and 3, but not 1, 2 and 3).

In each case a)– e) determine which order of games maximizes the probability of the event. In case e) assume in addition that $p + q > 3pq$.

2/II/10F **Probability**

A random point is distributed uniformly in a unit circle $\mathcal{D}$ so that the probability that it falls within a subset $\mathcal{A} \subseteq \mathcal{D}$ is proportional to the area of $\mathcal{A}$. Let R denote the distance between the point and the centre of the circle. Find the distribution function $F_R(x) = P(R < x)$, the expected value ER and the variance $\mathrm{Var}\, R = ER^2 - (ER)^2$.

Let Θ be the angle formed by the radius through the random point and the horizontal line. Prove that R and Θ are independent random variables.

Consider a coordinate system where the origin is placed at the centre of $\mathcal{D}$. Let X and Y denote the horizontal and vertical coordinates of the random point. Find the covariance $\mathrm{Cov}(X, Y) = E(XY) - EXEY$ and determine whether X and Y are independent.

Calculate the sum of expected values $E\frac{X}{R} + iE\frac{Y}{R}$. Show that it can be written as the expected value $Ee^{i\xi}$ and determine the random variable ξ.

2/II/11F Probability

Dipkomsky, a desperado in the wild West, is surrounded by an enemy gang and fighting tooth and nail for his survival. He has m guns, $m > 1$, pointing in different directions and tries to use them in succession to give an impression that there are several defenders. When he turns to a subsequent gun and discovers that the gun is loaded he fires it with probability 1/2 and moves to the next one. Otherwise, i.e. when the gun is unloaded, he loads it with probability 3/4 or simply moves to the next gun with complementary probability 1/4. If he decides to load the gun he then fires it or not with probability 1/2 and after that moves to the next gun anyway.

Initially, each gun had been loaded independently with probability p. Show that if after each move this distribution is preserved, then $p = 3/7$. Calculate the expected value EN and variance Var N of the number N of loaded guns under this distribution.

[*Hint: it may be helpful to represent N as a sum $\sum_{1\le j\le m} X_j$ of random variables taking values* 0 *and* 1.]

2/II/12F Probability

A taxi travels between four villages, W, X, Y, Z, situated at the corners of a rectangle. The four roads connecting the villages follow the sides of the rectangle; the distance from W to X and Y to Z is 5 miles and from W to Z and Y to X 10 miles. After delivering a customer the taxi waits until the next call then goes to pick up the new customer and takes him to his destination. The calls may come from any of the villages with probability 1/4 and each customer goes to any other village with probability 1/3. Naturally, when travelling between a pair of adjacent corners of the rectangle, the taxi takes the straight route, otherwise (when it travels from W to Y or X to Z or vice versa) it does not matter. Distances within a given village are negligible. Let D be the distance travelled to pick up and deliver a single customer. Find the probabilitites that D takes each of its possible values. Find the expected value ED and the variance Var D.

3/I/3C Vector Calculus

For a real function $f(x, y)$ with $x = x(t)$ and $y = y(t)$ state the chain rule for the derivative $\frac{d}{dt}f(x(t), y(t))$.

By changing variables to u and v, where $u = \alpha(x)y$ and $v = y/x$ with a suitable function $\alpha(x)$ to be determined, find the general solution of the equation

$$x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y} = 6f \ .$$

3/I/4A Vector Calculus

Suppose that

$$u = y^2 \sin(xz) + xy^2 z \cos(xz), \quad v = 2xy \sin(xz), \quad w = x^2 y^2 \cos(xz).$$

Show that $u\,dx + v\,dy + w\,dz$ is an exact differential.

Show that

$$\int_{(0,0,0)}^{(\pi/2,1,1)} u\,dx + v\,dy + w\,dz = \frac{\pi}{2}.$$

3/II/9C Vector Calculus

Explain, with justification, how the nature of a critical (stationary) point of a function $f(\mathbf{x})$ can be determined by consideration of the eigenvalues of the Hessian matrix H of $f(\mathbf{x})$ if H is non-singular. What happens if H is singular?

Let $f(x, y) = (y - x^2)(y - 2x^2) + \alpha x^2$. Find the critical points of f and determine their nature in the different cases that arise according to the values of the parameter $\alpha \in \mathbb{R}$.

3/II/10A Vector Calculus

State the rule for changing variables in a double integral.

Let D be the region defined by

$$\begin{cases} 1/x \le y \le 4x & \text{when } \frac{1}{2} \le x \le 1, \\ x \le y \le 4/x & \text{when } 1 \le x \le 2. \end{cases}$$

Using the transformation $u = y/x$ and $v = xy$, show that

$$\int_D \frac{4xy^3}{x^2+y^2}\, dxdy \;=\; \frac{15}{2} \ln \frac{17}{2}.$$

3/II/11B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{r})$ in a closed region V bounded by a smooth surface S.

Let $\Omega(\mathbf{r})$ be a scalar field. By choosing $\mathbf{u} = \mathbf{c}\,\Omega$ for arbitrary constant vector $\mathbf{c}$, show that

$$\int_V \nabla\,\Omega\, dv \;=\; \int_S \Omega\, d\mathbf{S}\ . \qquad (*)$$

Let V be the bounded region enclosed by the surface S which consists of the cone $(x, y, z) = (r\cos\theta, r\sin\theta, r/\sqrt{3})$ with $0 \le r \le \sqrt{3}$ and the plane $z = 1$, where r, θ, z are cylindrical polar coordinates. Verify that $(*)$ holds for the scalar field $\Omega = (a - z)$ where a is a constant.

3/II/12B Vector Calculus

In $\mathbb{R}^3$ show that, within a closed surface S, there is at most one solution of Poisson's equation, $\nabla^2\phi = \rho$, satisfying the boundary condition on S

$$\alpha\,\frac{\partial\phi}{\partial n} + \phi = \gamma\ ,$$

where α and γ are functions of position on S, and α is everywhere non-negative.

Show that

$$\phi(x, y) = e^{\pm lx}\,\sin ly$$

are solutions of Laplace's equation $\nabla^2\phi = 0$ on $\mathbb{R}^2$.

Find a solution $\phi(x, y)$ of Laplace's equation in the region $0 < x < \pi$, $0 < y < \pi$ that satisfies the boundary conditions

$$\begin{array}{cccc} \phi = 0 & \text{on} & 0 < x < \pi & y = 0 \\ \phi = 0 & \text{on} & 0 < x < \pi & y = \pi \\ \phi + \partial\phi/\partial n = 0 & \text{on} & x = 0 & 0 < y < \pi \\ \phi = \sin(ky) & \text{on} & x = \pi & 0 < y < \pi \end{array}$$

where k is a positive integer. Is your solution the only possible solution?