UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Wednesday 6 June 2001 9.00 to 12.00

PAPER 3

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, ..., L*** *according to the letter affixed to each question. (For example,* ***5C, 7C*** *should be in one bundle and* ***1D, 13D*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Explain what is meant by the *transition semigroup* $\{P_t\}$ of a Markov chain X in continuous time. If the state space is finite, show under assumptions to be stated clearly, that $P_t' = GP_t$ for some matrix G. Show that a distribution π satisfies $\pi G = 0$ if and only if $\pi P_t = \pi$ for all $t \geqslant 0$, and explain the importance of such π.

(ii) Let X be a continuous-time Markov chain on the state space $S = \{1, 2\}$ with generator

$$G = \begin{pmatrix} -\beta & \beta \\ \gamma & -\gamma \end{pmatrix}, \qquad \text{where} \quad \beta, \gamma > 0.$$

Show that the transition semigroup $P_t = \exp(tG)$ is given by

$$(\beta + \gamma)P_t = \begin{pmatrix} \gamma + \beta h(t) & \beta(1 - h(t)) \\ \gamma(1 - h(t)) & \beta + \gamma h(t) \end{pmatrix},$$

where $h(t) = e^{-t(\beta+\gamma)}$.

For $0 < \alpha < 1$, let

$$H(\alpha) = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}.$$

For a continuous-time chain X, let M be a matrix with (i, j) entry $P(X(1) = j \mid X(0) = i)$, for $i, j \in S$. Show that there is a chain X with $M = H(\alpha)$ if and only if $\alpha > \frac{1}{2}$.

2A Functional Analysis

(i) Define the notion of a measurable function between measurable spaces. Show that a continuous function $\mathbb{R}^2 \to \mathbb{R}$ is measurable with respect to the Borel σ-fields on $\mathbb{R}^2$ and $\mathbb{R}$.

By using this, or otherwise, show that, when $f, g : X \to \mathbb{R}$ are measurable with respect to some σ-field $\mathcal{F}$ on X and the Borel σ-field on $\mathbb{R}$, then $f + g$ is also measurable.

(ii) State the Monotone Convergence Theorem for $[0, \infty]$-valued functions. Prove the Dominated Convergence Theorem.

[*You may assume the Monotone Convergence Theorem but any other results about integration that you use will need to be stated carefully and proved.*]

Let X be the real Banach space of continuous real-valued functions on $[0, 1]$ with the uniform norm. Fix $u \in X$ and define

$$T : X \to \mathbb{R} \; ; \quad f \mapsto \int_0^1 f(t)u(t)\, dt \; .$$

Show that T is a bounded, linear map with norm

$$||T|| = \int_0^1 |u(t)|\, dt \; .$$

Is it true, for every choice of u, that there is function $f \in X$ with $||f|| = 1$ and $||T(f)|| = ||T||$?

3J Electromagnetism

(i) Develop the theory of electromagnetic waves starting from Maxwell equations in vacuum. You should relate the wave-speed c to ϵ_0 and μ_0 and establish the existence of plane, plane-polarized waves in which $\mathbf{E}$ takes the form

$$\mathbf{E} = (E_0 \cos(kz - \omega t), 0, 0) \; .$$

You should give the form of the magnetic field $\mathbf{B}$ in this case.

(ii) Starting from Maxwell's equation, establish Poynting's theorem.

$$-\mathbf{j} \cdot \mathbf{E} \;\; = \;\; \frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} \; ,$$

where $W = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$ and $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B}$. Give physical interpretations of W, S and the theorem.

Compute the averages over space and time of W and $\mathbf{S}$ for the plane wave described in (i) and relate them. Comment on the result.

4K Dynamics of Differential Equations

(i) Define a hyperbolic fixed point x_0 of a flow determined by a differential equation $\dot{x} = f(x)$ where $x \in R^n$ and f is C^1 (i.e. differentiable). State the Hartman-Grobman Theorem for flow near a hyperbolic fixed point. For nonlinear flows in R^2 with a hyperbolic fixed point x_0, does the theorem necessarily allow us to distinguish, on the basis of the linearized flow near x_0 between (a) a stable focus and a stable node; and (b) a saddle and a stable node? Justify your answers briefly.

(ii) Show that the system:

$$\begin{aligned}\dot{x} &= -(\mu+1) + (\mu-3)x - y + 6x^2 + 12xy + 5y^2 - 2x^3 - 6x^2y - 5xy^2, \\ \dot{y} &= 2 - 2x + (\mu-5)y + 4xy + 6y^2 - 2x^2y - 6xy^2 - 5y^3\end{aligned}$$

has a fixed point $(x_0, 0)$ on the x-axis. Show that there is a bifurcation at $\mu = 0$ and determine the stability of the fixed point for $\mu > 0$ and for $\mu < 0$.

Make a linear change of variables of the form $u = x - x_0 + \alpha y$, $v = x - x_0 + \beta y$, where α and β are constants to be determined, to bring the system into the form:

$$\begin{aligned}\dot{u} &= v + u[\mu - (u^2 + v^2)] \\ \dot{v} &= -u + v[\mu - (u^2 + v^2)]\end{aligned}$$

and hence determine whether the periodic orbit produced in the bifurcation is stable or unstable, and whether it exists in $\mu < 0$ or $\mu > 0$.

5C Representation Theory

Let $G = SU_2$, and V_n be the vector space of homogeneous polynomials of degree n in the variables x and y.

(i) Define the action of G on V_n, and prove that V_n is an irreducible representation of G.

(ii) Decompose $V_4 \otimes V_3$ into irreducible representations of SU_2. Briefly justify your answer.

(iii) SU_2 acts on the vector space $M_3(\mathbf{C})$ of complex 3×3 matrices via

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} X \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}, \qquad X \in M_3(\mathbf{C}).$$

Decompose this representation into irreducible representations.

6C Galois Theory

Let $\mathbf{F}_p$ be the finite field with p elements (p a prime), and let k be a finite extension of $\mathbf{F}_p$. Define the Frobenius automorphism $\sigma : k \longrightarrow k$, verifying that it is an $\mathbf{F}_p$-automorphism of k.

Suppose $f = X^{p+1} + X^p + 1 \in \mathbf{F}_p[X]$ and that K is its splitting field over $\mathbf{F}_p$. Why are the zeros of f distinct? If α is any zero of f in K, show that $\sigma(\alpha) = -\frac{1}{\alpha+1}$. Prove that f has at most two zeros in $\mathbf{F}_p$ and that $\sigma^3 = id$. Deduce that the Galois group of f over $\mathbf{F}_p$ is a cyclic group of order three.

7C Algebraic Topology

Write down the Mayer-Vietoris sequence and describe all the maps involved.

Use the Mayer-Vietoris sequence to compute the homology of the n-sphere S^n for all n.

8A Hilbert Spaces

Let T be a bounded linear operator on a Hilbert space H. Define what it means to say that T is (i) *compact*, and (ii) *Fredholm*. What is the *index*, $\operatorname{ind}(T)$, of a Fredholm operator T?

Let S, T be bounded linear operators on H. Prove that S and T are Fredholm if and only if both ST and TS are Fredholm. Prove also that if S is invertible and T is Fredholm then $\operatorname{ind}(ST) = \operatorname{ind}(TS) = \operatorname{ind}(T)$.

Let K be a compact linear operator on H. Prove that $I + K$ is Fredholm with index zero.

9B Riemann Surfaces

Let $f : X \to Y$ be a nonconstant holomorphic map between compact connected Riemann surfaces. Define the *valency* of f at a point, and the *degree* of f.

Define the *genus* of a compact connected Riemann surface X (assuming the existence of a triangulation).

State the Riemann-Hurwitz theorem. Show that a holomorphic non-constant self-map of a compact Riemann surface of genus $g > 1$ is bijective, with holomorphic inverse. Verify that the Riemann surface in $\mathbb{C}^2$ described in the equation $w^4 = z^4 - 1$ is non-singular, and describe its topological type.

[*Note: The description can be in the form of a picture or in words. If you apply Riemann-Hurwitz, explain first how you compactify the surface.*]

10B Algebraic Curves

Let C be the projective curve (over an algebraically closed field k of characteristic zero) defined by the affine equation

$$x^5 + y^5 = 1 \ .$$

Determine the points at infinity of C and show that C is smooth.

Determine the divisors of the rational functions $x, y \in k(C)$.

Show that $\omega = dx/y^4$ is a regular differential on C.

Compute the divisor of ω. What is the genus of C?

11B Logic, Computation and Set Theory

(i) Write down a set of axioms for the theory of dense linear order with a bottom element but no top element.

(ii) Prove that this theory has, up to isomorphism, precisely one countable model.

12D Probability and Measure

State and prove Birkhoff's almost-everywhere ergodic theorem.

[*You need not prove convergence in L_p and the maximal ergodic lemma may be assumed provided that it is clearly stated.*]

Let $\Omega = [0, 1)$, $\mathcal{F} = \mathcal{B}([0, 1))$ be the Borel σ-field and let $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. Give an example of an ergodic measure-preserving map $\theta : \Omega \to \Omega$ (you need not prove it is ergodic).

Let $f(x) = x$ for $x \in [0, 1)$. Find (at least for all x outside a set of measure zero)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (f \circ \theta^{i-1})(x).$$

Briefly justify your answer.

13D Applied Probability

Consider an $M/G/1$ queue with arrival rate λ and traffic intensity less than 1. Prove that the moment-generating function of a typical busy period, $M_B(\theta)$, satisfies

$$M_B(\theta) = M_S(\theta - \lambda + \lambda\ M_B(\theta)),$$

where $M_S(\theta)$ is the moment-generating function of a typical service time.

If service times are exponentially distributed with parameter $\mu > \lambda$, show that

$$M_B(\theta) = \frac{\lambda + \mu - \theta - \{(\lambda + \mu - \theta)^2 - 4\lambda\mu\ \}^{1/2}}{2\lambda}$$

for all sufficiently small but positive values of θ.

14D Optimization and Control

A file of X Mb is to be transmitted over a communications link. At each time t the sender can choose a transmission rate, $u(t)$, within the range $[0, 1]$ Mb per second. The charge for transmitting at rate $u(t)$ at time t is $u(t)p(t)$. The function p is fully known at time 0. If it takes a total time T to transmit the file then there is a delay cost of γT^2, $\gamma > 0$. Thus u and T are to be chosen to minimize

$$\int_0^T u(t)p(t)dt + \gamma T^2\,,$$

where $u(t) \in [0, 1]$, $dx(t)/dt = -u(t)$, $x(0) = X$ and $x(T) = 0$. Quoting and applying appropriate results of Pontryagin's maximum principle show that a property of the optimal policy is that there exists p^* such that $u(t) = 1$ if $p(t) < p^*$ and $u(t) = 0$ if $p(t) > p^*$.

Show that the optimal p^* and T are related by $p^* = p(T) + 2\gamma T$.

Suppose $p(t) = t + 1/t$ and $X = 1$. For what value of γ is it optimal to transmit at a constant rate 1 between times 1/2 and 3/2?

15E Principles of Statistics

(i) Explain what is meant by a *uniformly most powerful unbiased test* of a null hypothesis against an alternative.

Let $Y_1, \ldots, Y_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, with σ^2 known. Explain how to construct a uniformly most powerful unbiased size α test of the null hypothesis that $\mu = 0$ against the alternative that $\mu \neq 0$.

(ii) Outline briefly the Bayesian approach to hypothesis testing based on *Bayes factors*.

Let the distribution of $Y_1, \ldots, Y_n$ be as in (i) above, and suppose we wish to test, as in (i), $\mu = 0$ against the alternative $\mu \neq 0$. Suppose we assume a $N(0, \tau^2)$ prior for μ under the alternative. Find the form of the Bayes factor B, and show that, for fixed n, $B \to \infty$ as $\tau \to \infty$.

16D Stochastic Financial Models

(i) Suppose that Z is a random variable having the normal distribution with $\mathbb{E}Z = \beta$ and $\operatorname{Var} Z = \tau^2$.

For positive constants a, c show that

$$\mathbb{E}\left(ae^Z - c\right)_+ = ae^{\left(\beta + \tau^2/2\right)}\Phi\left(\frac{\log(a/c) + \beta}{\tau} + \tau\right) - c\Phi\left(\frac{\log(a/c) + \beta}{\tau}\right),$$

where Φ is the standard normal distribution function.

In the context of the Black-Scholes model, derive the formula for the price at time 0 of a European call option on the stock at strike price c and maturity time t_0 when the interest rate is ρ and the volatility of the stock is σ.

(ii) Let p denote the price of the call option in the Black-Scholes model specified in (i). Show that $\frac{\partial p}{\partial \rho} > 0$ and sketch carefully the dependence of p on the volatility σ (when the other parameters in the model are held fixed).

Now suppose that it is observed that the interest rate lies in the range $0 < \rho < \rho_0$ and when it changes it is linked to the volatility by the formula $\sigma = \ln\left(\rho_0/\rho\right)$. Consider a call option at strike price $c = S_0$, where S_0 is the stock price at time 0. There is a small increase $\Delta\rho$ in the interest rate; will the price of the option increase or decrease (assuming that the stock price is unaffected)? Justify your answer carefully.

[*You may assume that the function $\phi(x)/\Phi(x)$ is decreasing in x, $-\infty < x < \infty$, and increases to $+\infty$ as $x \to -\infty$, where Φ is the standard-normal distribution function and $\phi = \Phi'$.*]

17K Dynamical Systems

If $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ show that $A^{n+2} = A^{n+1} + A^n$ for all $n \geqslant 0$. Show that A^5 has trace 11 and deduce that the subshift map defined by A has just two cycles of exact period 5. What are they?

18A Partial Differential Equations

Write down a formula for the solution $u = u(t, x)$, for $t > 0$ and $x \in \mathbb{R}^n$, of the initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad u(0, x) = f(x),$$

for f a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. State (without proof) a theorem which ensures that this formula is the unique solution in some class of functions (which should be explicitly described).

By writing $u = e^t v$, or otherwise, solve the initial value problem

$$\frac{\partial v}{\partial t} + v - \Delta v = 0, \qquad v(0, x) = g(x), \tag{$\dagger$}$$

for g a bounded continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and give a class of functions in which your solution is the unique one.

Hence, or otherwise, prove that for all $t > 0$:

$$\sup_{x \in \mathbb{R}^n} v(t, x) \leqslant \sup_{x \in \mathbb{R}^n} g(x)$$

and deduce that the solutions $v_1(t, x)$ and $v_2(t, x)$ of ($\dagger$) corresponding to initial values $g_1(x)$ and $g_2(x)$ satisfy, for $t > 0$,

$$\sup_{x \in \mathbb{R}^n} |v_1(t, x) - v_2(t, x)| \leqslant \sup_{x \in \mathbb{R}^n} |g_1(x) - g_2(x)|.$$

19L Methods of Mathematical Physics

Consider the integral

$$\int_0^\infty \frac{t^z \mathrm{e}^{-at}}{1+t}\, dt\,,$$

where t^z is the principal branch and a is a positive constant. State the region of the complex z–plane in which the integral defines a holomorphic function.

Show how the analytic continuation of this function can be obtained by means of an alternative integral representation using the Hankel contour.

Hence show that the analytic continuation is holomorphic except for simple poles at $z = -1, -2, \ldots$, and that the residue at $z = -n$ is

$$(-1)^{n-1} \sum_{r=0}^{n-1} \frac{a^r}{r!}\,.$$

20K Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is discretized by the finite-difference method

$$u_m^{n+1} - \frac{1}{2}(\mu-\alpha)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}(\mu+\alpha)(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

where $u_m^n \approx u(m\Delta x, n\Delta t), \mu = \Delta t/(\Delta x)^2$ and α is a constant. Derive the order of magnitude (as a power of Δx) of the local error for different choices of α.

(ii) Investigate the stability of the above finite-difference method for different values of α by the Fourier technique.

21F Foundations of Quantum Mechanics

(i) Write the Hamiltonian for the harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

in terms of creation and annihilation operators, defined by

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - i\frac{p}{m\omega}\right), \qquad a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x + i\frac{p}{m\omega}\right).$$

Obtain an expression for $[a^\dagger, a]$ by using the usual commutation relation between p and x. Deduce the quantized energy levels for this system.

(ii) Define the number operator, N, in terms of creation and annihilation operators, $a^\dagger$ and a. The normalized eigenvector of N with eigenvalue n is $|n\rangle$. Show that $n \geq 0$.

Determine $a|n\rangle$ and $a^\dagger|n\rangle$ in the basis defined by $\{|n\rangle\}$.

Show that

$$a^{\dagger m} a^m |n\rangle = \begin{cases} \frac{n!}{(n-m)!}|n\rangle, & m \leq n\,, \\ 0\,, & m > n\,. \end{cases}$$

Verify the relation

$$|0\rangle\langle 0| = \sum_{m=0} \frac{1}{m!}(-1)^m a^{\dagger m} a^m\,,$$

by considering the action of both sides of the equation on an arbitrary basis vector.

22F Statistical Physics

A system consists of N weakly interacting non-relativistic fermions, each of mass m, in a three-dimensional volume, V. Derive the Fermi-Dirac distribution

$$n(\epsilon) \; = \; KVg\frac{\epsilon^{1/2}}{\exp((\epsilon-\mu)/kT)+1} \; ,$$

where $n(\epsilon)d\epsilon$ is the number of particles with energy in $(\epsilon, \epsilon + d\epsilon)$ and $K = 2\pi(2m)^{3/2}/h^3$. Explain the physical significance of g.

Explain how the constant μ is determined by the number of particles N and write down expressions for N and the internal energy E in terms of $n(\epsilon)$.

Show that, in the limit $\kappa \equiv e^{-\mu/kT} \gg 1$,

$$N \; = \; \frac{V}{A\kappa}\left(1 - \frac{1}{2\sqrt{2}\kappa} + O\left(\frac{1}{\kappa^2}\right)\right) \; ,$$

where $A = h^3/g(2\pi mkT)^{3/2}$.

Show also that in this limit

$$E \; = \; \frac{3}{2}NkT\left(1 + \frac{1}{4\sqrt{2}\kappa} + O\left(\frac{1}{\kappa^2}\right)\right) \; .$$

Deduce that the condition $\kappa \gg 1$ implies that $AN/V \ll 1$. Discuss briefly whether or not this latter condition is satisfied in a white dwarf star and in a dilute electron gas at room temperature.

$\left[\textit{Note that } \int_0^\infty du\, e^{-u^2 a} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]$.

23J Applications of Quantum Mechanics

Write down the commutation relations satisfied by the cartesian components of the total angular momentum operator $\mathbf{J}$.

In quantum mechanics an operator $\mathbf{V}$ is said to be a vector operator if, under rotations, its components transform in the same way as those of the coordinate operator $\mathbf{r}$. Show from first principles that this implies that its cartesian components satisfy the commutation relations

$$[J_j, V_k] = i\epsilon_{jkl}V_l \; .$$

Hence calculate the total angular momentum of the nonvanishing states $V_j|0\rangle$, where $|0\rangle$ is the vacuum state.

24H Fluid Dynamics II

A planar flow of an inviscid, incompressible fluid is everywhere in the x-direction and has velocity profile

$$u = \begin{cases} U & y > 0, \\ 0 & y < 0. \end{cases}$$

By examining linear perturbations to the vortex sheet at $y = 0$ that have the form $e^{ikx - i\omega t}$, show that

$$\omega = \frac{1}{2} kU(1 \pm i)$$

and deduce the temporal stability of the sheet to disturbances of wave number k.

Use this result to determine also the spatial growth rate and propagation speed of disturbances of frequency ω introduced at a fixed spatial position.

25L Waves in Fluid and Solid Media

Consider the equation

$$\phi_{tt} + \alpha^2 \phi_{xxxx} + \beta^2 \phi = 0, \qquad (*)$$

where α and β are real constants. Find the dispersion relation for waves of frequency ω and wavenumber k. Find the phase velocity $c(k)$ and the group velocity $c_g(k)$ and sketch graphs of these functions.

Multiplying equation $(*)$ by ϕ_t, obtain an equation of the form

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} = 0$$

where A and B are expressions involving ϕ and its derivatives. Give a physical interpretation of this equation.

Evaluate the time-averaged energy $\langle E \rangle$ and energy flux $\langle I \rangle$ of a monochromatic wave $\phi = \cos(kx - wt)$, and show that

$$\langle I \rangle = c_g \langle E \rangle.$$