UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Tuesday 5 June 2001 9.00 to 12.00

PAPER 2

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only ***one*** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***2A, 5A*** *should be in one bundle and* ***1H, 24H*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1H Principles of Dynamics

(i) An axially symmetric top rotates freely about a fixed point O on its axis. The principal moments of inertia are A, A, C and the centre of gravity G is a distance h from O.

Define the three Euler angles θ, ϕ and ψ, specifying the orientation of the top. Use Lagrange's equations to show that there are three conserved quantities in the motion. Interpret them physically.

(ii) Initially the top is spinning with angular speed n about OG, with OG vertical, before it is slightly disturbed.

Show that, in the subsequent motion, θ stays close to zero if $C^2n^2 > 4mghA$, but if this condition fails then θ attains a maximum value given approximately by

$$\cos\theta \approx \frac{C^2n^2}{2mghA} - 1.$$

Why is this only an approximation?

2A Functional Analysis

(i) State the Stone-Weierstrass theorem for complex-valued functions. Use it to show that the trigonometric polynomials are dense in the space $C(\mathbb{T})$ of continuous, complex-valued functions on the unit circle $\mathbb{T}$ with the uniform norm.

Show further that, for $f \in C(\mathbb{T})$, the nth Fourier coefficient

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta$$

tends to 0 as $|n|$ tends to infinity.

(ii) **(a)** Let X be a normed space with the property that the series $\sum_{n=1}^{\infty} x_n$ converges whenever (x_n) is a sequence in X with $\sum_{n=1}^{\infty} ||x_n||$ convergent. Show that X is a Banach space.

(b) Let K be a compact metric space and L a closed subset of K. Let $R : C(K) \to C(L)$ be the map sending $f \in C(K)$ to its restriction $R(f) = f|L$ to L. Show that R is a bounded, linear map and that its image is a subalgebra of $C(L)$ separating the points of L.

Show further that, for each function g in the image of R, there is a function $f \in C(K)$ with $R(f) = g$ and $||f||_\infty = ||g||_\infty$. Deduce that every continuous, complex-valued function on L can be extended to a continuous function on all of K.

3C Groups, Rings and Fields

(i) Show that the ring $k = \mathbf{F}_2[X]/(X^2 + X + 1)$ is a field. How many elements does it have?

(ii) Let k be as in (i). By considering what happens to a chosen basis of the vector space k^2, or otherwise, find the order of the groups $GL_2(k)$ and $SL_2(k)$.

By considering the set of lines in k^2, or otherwise, show that $SL_2(k)$ is a subgroup of the symmetric group S_5, and identify this subgroup.

4K Dynamics of Differential Equations

(i) Define a Liapounov function for a flow ϕ on $\mathbb{R}^n$. Explain what it means for a fixed point of the flow to be Liapounov stable. State and prove Liapounov's first stability theorem.

(ii) Consider the damped pendulum

$$\ddot{\theta} + k\dot{\theta} + \sin\theta = 0,$$

where $k > 0$. Show that there are just two fixed points (considering the phase space as an infinite cylinder), and that one of these is the origin and is Liapounov stable. Show further that the origin is asymptotically stable, and that the the ω-limit set of each point in the phase space is one or other of the two fixed points, justifying your answer carefully.

[*You should state carefully any theorems you use in your answer, but you need not prove them.*]

5A Combinatorics

As usual, $R_k^{(r)}(m)$ denotes the smallest integer n such that every k-colouring of $[n]^{(r)}$ yields a monochromatic m-subset $M \in [n]^{(m)}$. Prove that $R_2^{(2)}(m) > 2^{m/2}$ for $m \geqslant 3$.

Let $\mathcal{P}([n])$ have the colex order, and for $a, b \in \mathcal{P}([n])$ let $\delta(a,b) = \max a \triangle b$; thus $a < b$ means $\delta(a,b) \in b$. Show that if $a < b < c$ then $\delta(a,b) \neq \delta(b,c)$, and that $\delta(a,c) = \max\{\delta(a,b), \delta(b,c)\}$.

Given a red-blue colouring of $[n]^{(2)}$, the 4-colouring

$$\chi : \mathcal{P}([n])^{(3)} \to \{\text{red}, \text{blue}\} \times \{0,1\}$$

is defined as follows:

$$\chi(\{a,b,c\}) = \begin{cases} (\text{red}, 0) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is red and } \delta(a,b) < \delta(b,c) \\ (\text{red}, 1) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is red and } \delta(a,b) > \delta(b,c) \\ (\text{blue}, 0) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is blue and } \delta(a,b) < \delta(b,c) \\ (\text{blue}, 1) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is blue and } \delta(a,b) > \delta(b,c) \end{cases}$$

where $a < b < c$. Show that if $M = \{a_0, a_1, \ldots, a_m\} \in \mathcal{P}([n])^{(m+1)}$ is monochromatic then $\{\delta_1, \ldots, \delta_m\} \in [n]^{(m)}$ is monochromatic, where $\delta_i = \delta(a_{i-1}, a_i)$ and $a_0 < a_1 < \cdots < a_m$.

Deduce that $R_4^{(3)}(m+1) > 2^{2^{m/2}}$ for $m \geqslant 3$.

6C Representation Theory

(i) Let G be a group, and X and Y finite G-sets. Define the permutation representation $\mathbf{C}[X]$ and compute its character. Show that

$$\dim \operatorname{Hom}_G(\mathbf{C}[X], \mathbf{C}[Y])$$

is equal to the number of G-orbits in $X \times Y$.

(ii) Let $G = S_n$ $(n \geqslant 4)$, $X = \{1, \ldots, n\}$, and

$$Z = \{\, \{i,j\} \subseteq X \mid i \neq j \}$$

be the set of 2-element subsets of X. Decompose $\mathbf{C}[Z]$ into irreducibles, and determine the dimension of each irreducible constituent.

7B Differentiable Manifolds

State Stokes' Theorem.

Prove that, if M^m is a compact connected manifold and $\Phi : U \to \mathbb{R}^m$ is a surjective chart on M, then for any $\omega \in \Omega^m(M)$ there is $\eta \in \Omega^{m-1}(M)$ such that $\mathrm{supp}(\omega + d\eta) \subseteq \Phi^{-1}(\mathbf{B}^m)$, where $\mathbf{B}^m$ is the unit ball in $\mathbb{R}^m$.

[*You may assume that, if* $\omega \in \Omega^m(\mathbb{R}^m)$ *with* $\mathrm{supp}(\omega) \subseteq \mathbf{B}^m$ *and* $\int\limits_{\mathbb{R}^m} \omega = 0$*, then* $\exists\, \eta \in \Omega^{m-1}(\mathbb{R}^m)$ *with* $\mathrm{supp}(\eta) \subseteq \mathbf{B}^m$ *such that* $d\eta = \omega$*.*]

By considering the m-form

$$\omega = x_1 dx_2 \wedge ... \wedge dx_{m+1} + \cdots + x_{m+1} dx_1 \wedge ... \wedge dx_m$$

on $\mathbb{R}^{m+1}$, or otherwise, deduce that $H^m(S^m) \cong \mathbb{R}$.

8C Algebraic Topology

Show that the fundamental group of the 2-torus $S^1 \times S^1$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

Show that an injective continuous map from the circle S^1 to itself induces multiplication by ± 1 on the fundamental group.

Show that there is no retraction from the solid torus $S^1 \times D^2$ to its boundary.

9B Number Fields

Determine the ideal class group of $\mathbf{Q}(\sqrt{-11})$.

Find all solutions of the diophantine equation

$$y^2 + 11 = x^3 \qquad (x, y \in \mathbf{Z})\ .$$

[*Minkowski's bound is* $n! n^{-n} (4/\pi)^{r_2} |D_k|^{1/2}$*.*]

10B Algebraic Curves

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map given by $f(X_0 : X_1 : X_2) = (X_1X_2 : X_0X_2 : X_0X_1)$. Determine whether f is defined at the following points: $(1:1:1), (0:1:1), (0:0:1)$.

Let $C \subset \mathbb{P}^2$ be the curve defined by $X_1^2 X_2 - X_0^3 = 0$. Define a bijective morphism $\alpha : \mathbb{P}^1 \to C$. Prove that α is not an isomorphism.

11B Logic, Computation and Set Theory

Let U be an arbitrary set, and $\mathcal{P}(U)$ the power set of U. For X a subset of $\mathcal{P}(U)$, the *dual* $X^\vee$ of X is the set $\{y \subseteq U : (\forall x \in X)(y \cap x \neq \emptyset)\}$.

(i) Show that $X \subseteq Y \to Y^\vee \subseteq X^\vee$.

Show that for $\{X_i : i \in I\}$ a family of subsets of $\mathcal{P}(U)$

$$\left(\bigcup\{X_i : i \in I\}\right)^\vee = \bigcap\{X_i^\vee : i \in I\}.$$

(ii) Consider $S = \{X \subseteq \mathcal{P}(U) : X \subseteq X^\vee\}$. Show that $S, \subseteq$ is a chain-complete poset.

State Zorn's lemma and use it to deduce that there exists X with $X = X^\vee$.

Show that if $X = X^\vee$ then the following hold:

X is closed under superset; for all $U' \subseteq U$, X contains either U' or $U \setminus U'$.

12D Probability and Measure

(a) Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}((0,1))$ be the Borel σ-field and let $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. What is the distribution of the random variable Z, where $Z(\omega) = 2\omega - 1$?

Let $\omega = \sum_{n=1}^{\infty} 2^{-n} R_n(\omega)$ be the binary expansion of the point $\omega \in \Omega$ and set $U(\omega) = \sum\limits_{n\ odd} 2^{-n} Q_n(\omega)$, where $Q_n(\omega) = 2R_n(\omega) - 1$. Find a random variable V independent of U such that U and V are identically distributed and $U + \frac{1}{2}V$ is uniformly distributed on $(-1,1)$.

(b) Now suppose that on some probability triple X and Y are independent, identically-distributed random variables such that $X + \frac{1}{2}Y$ is uniformly distributed on $(-1,1)$.

Let ϕ be the characteristic function of X. Calculate $\phi(t)/\phi(t/4)$. Show that the distribution of X must be the same as the distribution of the random variable U in (a).

13D Applied Probability

Let M be a Poisson random measure on $E = \mathbb{R} \times [0,\pi)$ with constant intensity λ. For $(x,\theta) \in E$, denote by $l(x,\theta)$ the line in $\mathbb{R}^2$ obtained by rotating the line $\{(x,y) : y \in \mathbb{R}\}$ through an angle θ about the origin.

Consider the line process $L = M \circ\ l^{-1}$.

(i) What is the distribution of the number of lines intersecting the disc $\{z \in \mathbb{R}^2 : |z| \leqslant a\}$?

(ii) What is the distribution of the distance from the origin to the nearest line?

(iii) What is the distribution of the distance from the origin to the kth nearest line?

14E Information Theory

A subset $\mathcal{C}$ of the Hamming space $\{0,1\}^n$ of cardinality $|\mathcal{C}| = r$ and with the minimal (Hamming) distance $\min\,[d(x,x') : x, x' \in \mathcal{C}, x \neq x'] = \delta$ is called an $[n,r,\delta]$-code (not necessarily linear). An $[n,r,\delta]$-code is called *maximal* if it is not contained in any $[n, r+1, \delta]$-code. Prove that an $[n,r,\delta]$-code is maximal if and only if for any $y \in \{0,1\}^n$ there exists $x \in \mathcal{C}$ such that $d(x,y) < \delta$. Conclude that if there are δ or more changes made in a codeword then the new word is closer to some other codeword than to the original one.

Suppose that a maximal $[n,r,\delta]$-code is used for transmitting information via a binary memoryless channel with the error probability p, and the receiver uses the maximum likelihood decoder. Prove that the probability of erroneous decoding, $\pi_{\mathrm{err}}^{\mathrm{ml}}$, obeys the bounds

$$1 - b(n, \delta - 1) \leqslant \pi_{\mathrm{err}}^{\mathrm{ml}} \leqslant 1 - b(n, [(\delta-1)/2]),$$

where

$$b(n,m) = \sum_{0 \leqslant k \leqslant m} \binom{n}{k} p^k (1-p)^{n-k}$$

is a partial binomial sum and $[\,\cdot\,]$ is the integer part.

15D Optimization and Control

A street trader wishes to dispose of k counterfeit Swiss watches. If he offers one for sale at price u he will sell it with probability ae^{-u}. Here a is known and less than 1. Subsequent to each attempted sale (successful or not) there is a probability $1-\beta$ that he will be arrested and can make no more sales. His aim is to choose the prices at which he offers the watches so as to maximize the expected values of his sales up until the time he is arrested or has sold all k watches.

Let $V(k)$ be the maximum expected amount he can obtain when he has k watches remaining and has not yet been arrested. Explain why $V(k)$ is the solution to

$$V(k) = \max_{u>0} \left\{ ae^{-u}[u + \beta V(k-1)] + (1 - ae^{-u})\beta V(k) \right\}\,.$$

Denote the optimal price by u_k and show that

$$u_k = 1 + \beta V(k) - \beta V(k-1)$$

and that

$$V(k) = ae^{-u_k}/(1-\beta)\,.$$

Show inductively that $V(k)$ is a nondecreasing and concave function of k.

16E Principles of Statistics

(i) Let $X_1, \ldots, X_n$ be independent, identically-distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a minimal sufficient statistic for μ.

Let $T_1 = n^{-1}\sum_{i=1}^n X_i$ and $T_2 = \sqrt{n^{-1}\sum_{i=1}^n X_i^2}$. Write down the distribution of X_i/μ, and hence show that $Z = T_1/T_2$ is ancillary. Explain briefly why the Conditionality Principle would lead to inference about μ being drawn from the conditional distribution of T_2 given Z.

What is the maximum likelihood estimator of μ?

(ii) Describe briefly the Bayesian approach to predictive inference.

Let $Z_1, \ldots, Z_n$ be independent, identically-distributed $N(\mu, \sigma^2)$ random variables, with μ, σ^2 both unknown. Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2 based on $Z_1, \ldots, Z_n$, and state, without proof, their joint distribution.

Suppose that it is required to construct a prediction interval $I_{1-\alpha} \equiv I_{1-\alpha}(Z_1, \ldots, Z_n)$ for a future, independent, random variable Z_0 with the same $N(\mu, \sigma^2)$ distribution, such that

$$P(Z_0 \in I_{1-\alpha}) = 1 - \alpha,$$

with the probability over the *joint* distribution of $Z_0, Z_1, \ldots, Z_n$. Let

$$I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2) = \left[\bar{Z}_n - z_{\alpha/2}\sigma\sqrt{1 + 1/n},\ \bar{Z}_n + z_{\alpha/2}\sigma\sqrt{1 + 1/n}\right],$$

where $\bar{Z}_n = n^{-1}\sum_{i=1}^n Z_i$, and $\Phi(z_\beta) = 1 - \beta$, with Φ the distribution function of $N(0,1)$.

Show that $P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2)) = 1 - \alpha$.

By considering the distribution of $(Z_0 - \bar{Z}_n)/\left(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}}\right)$, or otherwise, show that

$$P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) < 1 - \alpha,$$

and show how to construct an interval $I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)$ with

$$P(Z_0 \in I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) = 1 - \alpha.$$

[*Hint: if Y has the t-distribution with m degrees of freedom and $t_\beta^{(m)}$ is defined by $P(Y < t_\beta^{(m)}) = 1 - \beta$ then $t_\beta > z_\beta$ for $\beta < \frac{1}{2}$.*]

17A Partial Differential Equations

Define the Schwartz space $\mathcal{S}(\mathbb{R})$ and the corresponding space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

Use the Fourier transform to give an integral formula for the solution of the equation

$$-\frac{d^2u}{dx^2}+\frac{du}{dx}+u=f \qquad (*)$$

for $f\in\mathcal{S}(\mathbb{R})$. Prove that your solution lies in $\mathcal{S}(\mathbb{R})$. Is your formula the unique solution to $(*)$ in the Schwartz space?

Deduce from this formula an integral expression for the fundamental solution of the operator $P=-\frac{d^2}{dx^2}+\frac{d}{dx}+1$.

Let K be the function:

$$K(x)=\begin{cases}\frac{1}{\sqrt{5}}e^{-(\sqrt{5}-1)x/2} & \text{for } x\geqslant 0,\\ \frac{1}{\sqrt{5}}e^{(\sqrt{5}+1)x/2} & \text{for } x\leqslant 0.\end{cases}$$

Using the definition of distributional derivatives verify that this function is a fundamental solution for P.

18L Methods of Mathematical Physics

The Bessel function $J_\nu(z)$ is defined, for $|\arg z|<\pi/2$, by

$$J_\nu(z)=\frac{1}{2\pi i}\int_{-\infty}^{(0^+)}\mathrm{e}^{(t-t^{-1})z/2}t^{-\nu-1}dt\ ,$$

where the path of integration is the Hankel contour and $t^{-\nu-1}$ is the principal branch.

Use the method of steepest descent to show that, as $z\to+\infty$,

$$J_\nu(z)\sim(2/\pi z)^{\frac{1}{2}}\cos(z-\pi\nu/2-\pi/4)\ .$$

You should give a rough sketch of the steepest descent paths.

19K Numerical Analysis

(i) Define m-step BDF (backward differential formula) methods for the numerical solution of ordinary differential equations and derive explicitly their coefficients.

(ii) Prove that the linear stability domain of the two-step BDF method includes the interval $(-\infty,0)$.

20F Electrodynamics

In a superconductor, there are superconducting charge carriers with number density n, mass m and charge q. Starting from the quantum mechanical wavefunction $\Psi = Re^{i\Phi}$ (with real R and Φ), construct a formula for the electric current and explain carefully why your result is gauge invariant.

Now show that inside a superconductor a static magnetic field obeys the equation

$$\nabla^2 \mathbf{B} = \frac{\mu_0 \, n \, q^2}{m} \mathbf{B}.$$

A superconductor occupies the region $z > 0$, while for $z < 0$ there is a vacuum with a constant magnetic field in the x direction. Show that the magnetic field cannot penetrate deep into the superconductor.

21F Foundations of Quantum Mechanics

(i) Hermitian operators $\hat{x}$, $\hat{p}$, satisfy $[\hat{x}, \hat{p}] = i\hbar$. The eigenvectors $|p\rangle$, satisfy $\hat{p}|p\rangle = p|p\rangle$ and $\langle p'|p\rangle = \delta(p'-p)$. By differentiating with respect to b verify that

$$e^{-ib\hat{x}/\hbar}\hat{p}\, e^{ib\hat{x}/\hbar} = \hat{p} + b$$

and hence show that

$$e^{ib\hat{x}/\hbar}|p\rangle = |p+b\rangle.$$

Show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

and

$$\langle p|\hat{p}|\psi\rangle = p\,\langle p|\psi\rangle\,.$$

(ii) A quantum system has Hamiltonian $H = H_0 + H_1$, where H_1 is a small perturbation. The eigenvalues of H_0 are ϵ_n. Give (without derivation) the formulae for the first order and second order perturbations in the energy level of a non-degenerate state. Suppose that the rth energy level of H_0 has j degenerate states. Explain how to determine the eigenvalues of H corresponding to these states to first order in H_1.

In a particular quantum system an orthonormal basis of states is given by $|n_1, n_2\rangle$, where n_i are integers. The Hamiltonian is given by

$$H = \sum_{n_1,n_2}(n_1^2+n_2^2)\,|n_1,n_2\rangle\langle n_1,n_2| + \sum_{n_1,n_2,n'_1,n'_2}\lambda_{|n_1-n'_1|,|n_2-n'_2|}\,|n_1,n_2\rangle\langle n'_1,n'_2|\,,$$

where $\lambda_{r,s} = \lambda_{s,r}$, $\lambda_{0,0} = 0$ and $\lambda_{r,s} = 0$ unless r and s are both even.

Obtain an expression for the ground state energy to second order in the perturbation, $\lambda_{r,s}$. Find the energy eigenvalues of the first excited state to first order in the perturbation. Determine a matrix (which depends on two independent parameters) whose eigenvalues give the first order energy shift of the second excited state.

22J Applications of Quantum Mechanics

A particle of charge e moves freely within a cubical box of side a. Its initial wavefunction is

$$(2/a)^{-\frac{3}{2}}\sin(\pi x/a)\sin(\pi y/a)\sin(\pi z/a).$$

A uniform electric field $\mathcal{E}$ in the x direction is switched on for a time T. Derive from first principles the probability, correct to order $\mathcal{E}^2$, that after the field has been switched off the wave function will be found to be

$$(2/a)^{-\frac{3}{2}}\sin(2\pi x/a)\sin(\pi y/a)\sin(\pi z/a).$$

23J General Relativity

(i) Show that the geodesic equation follows from a variational principle with Lagrangian

$$L \quad = \quad g_{ab}\dot{x}^a\dot{x}^b$$

where the path of the particle is $x^a(\lambda)$, and λ is an affine parameter along that path.

(ii) The Schwarzschild metric is given by

$$ds^2 = dr^2\left(1 - \frac{2M}{r}\right)^{-1} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2M}{r}\right)dt^2.$$

Consider a photon which moves within the equatorial plane $\theta = \frac{\pi}{2}$. Using the above Lagrangian, or otherwise, show that

$$\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right) = E, \quad \text{and} \quad r^2\left(\frac{d\phi}{d\lambda}\right) = h,$$

for constants E and h. Deduce that

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right). \qquad (*)$$

Assume further that the photon approaches from infinity. Show that the impact parameter b is given by

$$b = \frac{h}{E} \quad .$$

By considering the equation $(*)$, or otherwise

(a) show that, if $b^2 > 27M^2$, the photon is deflected but not captured by the black hole;

(b) show that, if $b^2 < 27M^2$, the photon is captured;

(c) describe, with justification, the qualitative form of the photon's orbit in the case $b^2 = 27M^2$.

24H Fluid Dynamics II

Explain what is meant by a Stokes flow and show that, in such a flow, in the absence of body forces, $\partial\sigma_{ij}/\partial x_j = 0$, where σ_{ij} is the stress tensor.

State and prove the *reciprocal theorem* for Stokes flow.

When a rigid sphere of radius a translates with velocity $\mathbf{U}$ through unbounded fluid at rest at infinity, it may be shown that the traction per unit area, $\boldsymbol{\sigma}\cdot\mathbf{n}$, exerted by the sphere on the fluid, has the uniform value $3\mu\mathbf{U}/2a$ over the sphere surface. Find the drag on the sphere.

Suppose that the same sphere is free of external forces and is placed with its centre at the origin in an unbounded Stokes flow given in the absence of the sphere as $\mathbf{u}_s(\mathbf{x})$. By applying the reciprocal theorem to the perturbation to the flow generated by the presence of the sphere, and assuming this to tend to zero sufficiently rapidly at infinity, show that the instantaneous velocity of the centre of the sphere is

$$\mathbf{V} = \frac{1}{4\pi a^2}\int_{r=a}\mathbf{u}_s(\mathbf{x})dS.$$

25L Waves in Fluid and Solid Media

A semi-infinite elastic medium with shear modulus μ_1 and shear-wave speed c_1 lies in $y < 0$. Above it there is a layer $0 \leq y \leqslant h$ of a second elastic medium with shear modulus μ_2 and shear-wave speed c_2 $(< c_1)$. The top boundary $y = h$ is stress-free. Consider a monochromatic shear wave propagating at speed c with wavenumber k in the x-direction and with displacements only in the z-direction.

Obtain the dispersion relation

$$\tan kh\theta = \frac{\mu_1 c_2}{\mu_2 c_1}\,\frac{1}{\theta}\left(\frac{c_1^2}{c_2^2} - 1 - \theta^2\right)^{1/2}, \qquad \text{where} \quad \theta = \sqrt{\frac{c^2}{c_2^2} - 1}.$$

Deduce that the modes have a cut-off frequency $\pi n c_1 c_2/h\sqrt{c_1^2 - c_2^2}$ where they propagate at speed $c = c_1$.