UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Thursday 7 June 2001 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than ***FOUR*** *questions.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***3A, 9A*** *should be in one bundle and* ***4C, 10C*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

Write an essay on the convergence to equilibrium of a discrete-time Markov chain on a countable state-space. You should include a discussion of the existence of invariant distributions, and of the limit theorem in the non-null recurrent case.

2H Principles of Dynamics

(i) Consider a particle of charge q and mass m, moving in a stationary magnetic field $\mathbf{B}$. Show that Lagrange's equations applied to the Lagrangian

$$L = \frac{1}{2} m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}),$$

where $\mathbf{A}$ is the vector potential such that $\mathbf{B} = \text{curl } \mathbf{A}$, lead to the correct Lorentz force law. Compute the canonical momentum $\mathbf{p}$, and show that the Hamiltonian is $H = \frac{1}{2} m\dot{\mathbf{r}}^2$.

(ii) Expressing the velocity components $\dot{r}_i$ in terms of the canonical momenta and co-ordinates for the above system, derive the following formulae for Poisson brackets:

(a) $\{FG, H\} = F\{G, H\} + \{F, H\} G$, for any functions F, G, H;

(b) $\{m\dot{r}_i, m\dot{r}_j\} = q\epsilon_{ijk} B_k$;

(c) $\{m\dot{r}_i, r_j\} = -\delta_{ij}$;

(d) $\{m\dot{r}_i, f(r_j)\} = -\frac{\partial}{\partial r_i} f(r_j)$.

Now consider a particle moving in the field of a magnetic monopole,

$$B_i = g\frac{r_i}{r^3}.$$

Show that $\{H, \mathbf{J}\} = 0$, where $\mathbf{J} = m\mathbf{r} \wedge \dot{\mathbf{r}} - gq\hat{\mathbf{r}}$. Explain why this means that $\mathbf{J}$ is conserved.

Show that, if $g = 0$, conservation of $\mathbf{J}$ implies that the particle moves in a plane perpendicular to $\mathbf{J}$. What type of surface does the particle move on if $g \neq 0$?

3A Functional Analysis

Write an account of the classical sequence spaces: ℓ_p $(1 \leqslant p \leqslant \infty)$ and c_0. You should define them, prove that they are Banach spaces, and discuss their properties, including their dual spaces. Show that ℓ_∞ is inseparable but that c_0 and ℓ_p for $1 \leqslant p < \infty$ are separable.

Prove that, if $T : X \to Y$ is an isomorphism between two Banach spaces, then

$$T^* : Y^* \to X^* \; ; \quad f \mapsto f \circ T$$

is an isomorphism between their duals.

Hence, or otherwise, show that no two of the spaces $c_0, \ell_1, \ell_2, \ell_\infty$ are isomorphic.

4C Groups, Rings and Fields

Show that the ring $\mathbf{Z}[\omega]$ is Euclidean, where $\omega = \exp(2\pi i/3)$.

Show that a prime number $p \neq 3$ is reducible in $\mathbf{Z}[\omega]$ if and only if $p \equiv 1 (\text{mod } 3)$.

Which prime numbers p can be written in the form $p = a^2 + ab + b^2$ with $a, b \in \mathbf{Z}$ (and why)?

5J Electromagnetism

Write down the form of Ohm's Law that applies to a conductor if at a point $\mathbf{r}$ it is moving with velocity $\mathbf{v}(\mathbf{r})$.

Use two of Maxwell's equations to prove that

$$\int_C (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot d\mathbf{r} \quad = \quad -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad ,$$

where $C(t)$ is a moving closed loop, $\mathbf{v}$ is the velocity at the point $\mathbf{r}$ on C, and S is a surface spanning C. The time derivative on the right hand side accounts for changes in both C and $\mathbf{B}$. Explain briefly the physical importance of this result.

Find and sketch the magnetic field $\mathbf{B}$ described in the vector potential

$$\mathbf{A}(r, \theta, z) \quad = \quad (0, \tfrac{1}{2}\, brz, 0)$$

in cylindrical polar coordinates (r, θ, z) , where $b > 0$ is constant.

A conducting circular loop of radius a and resistance R lies in the plane $z = h(t)$ with its centre on the z-axis.

Find the magnitude and direction of the current induced in the loop as $h(t)$ changes with time, neglecting self-inductance.

At time $t = 0$ the loop is at rest at $z = 0$. For time $t > 0$ the loop moves with constant velocity $dh/dt = v > 0$. Ignoring the inertia of the loop, use energy considerations to find the force $F(t)$ necessary to maintain this motion.

[*In cylindrical polar coordinates*

$$\text{curl } \mathbf{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r} \frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right). \Big]$$

6K Dynamics of Differential Equations

Write a short essay about periodic orbits in flows in two dimensions. Your essay should include criteria for the existence and non-existence of periodic orbits, and should mention (with sketches) at least two bifurcations that create or destroy periodic orbits in flows as a parameter is altered (though a detailed analysis of any bifurcation is not required).

7C Geometry of Surfaces

Write an essay on the Gauss-Bonnet theorem. Make sure that your essay contains a precise statement of the theorem, in its local form, and a discussion of some of its applications, including the global Gauss-Bonnet theorem.

8B Logic, Computation and Set Theory

What is a wellfounded relation, and how does wellfoundedness underpin wellfounded induction?

A formula $\phi(x, y)$ with two free variables *defines an* $\in$-*automorphism* if for all x there is a unique y, the function f, defined by $y = f(x)$ if and only if $\phi(x, y)$, is a permutation of the universe, and we have $(\forall xy)(x \in y \leftrightarrow f(x) \in f(y))$.

Use wellfounded induction over $\in$ to prove that all formulæ defining $\in$-automorphisms are equivalent to $x = y$.

9A Graph Theory

Write an essay on extremal graph theory. Your essay should include proofs of at least two major results and a discussion of variations on these results or their proofs.

10C Number Theory

Attempt **one** of the following:

(i) Discuss pseudoprimes and primality testing. Find all bases for which 57 is a Fermat pseudoprime. Determine whether 57 is also an Euler pseudoprime for these bases.

(ii) Write a brief account of various methods for factoring large numbers. Use Fermat factorization to find the factors of 10033. Would Pollard's $p - 1$ method also be practical in this instance?

(iii) Show that $\sum 1/p_n$ is divergent, where p_n denotes the n-*th* prime.

Write a brief account of basic properties of the Riemann zeta-function.

State the prime number theorem. Show that it implies that for all sufficiently large positive integers n there is a prime p satisfying $n < p \leqslant 2n$.

11E Algorithms and Networks

State the optimal distribution problem. Carefully describe the simplex-on-a-graph algorithm for solving optimal distribution problems when the flow in each arc in the network is constrained to lie in the interval $[0, \infty)$. Explain how the algorithm can be initialised if there is no obvious feasible solution with which to begin. Describe the adjustments that are needed for the algorithm to cope with more general capacity constraints $x(j) \in [c^-(j), c^+(j)]$ for each arc j (where $c^{\pm}(j)$ may be finite or infinite).

12D Stochastic Financial Models

Write an essay on the mean-variance approach to portfolio selection in a one-period model. Your essay should contrast the solution in the case when all the assets are risky with that for the case when there is a riskless asset.

13E Principles of Statistics

Write an account, with appropriate examples, of **one** of the following:

(a) Inference in multi-parameter exponential families;

(b) Asymptotic properties of maximum-likelihood estimators and their use in hypothesis testing;

(c) Bootstrap inference.

14E Computational Statistics and Statistical Modelling

(i) Assume that independent observations $Y_1, \ldots, Y_n$ are such that

$$Y_i \sim \text{Binomial}(t_i, \pi_i), \ \log \frac{\pi_i}{1-\pi_i} = \beta^T x_i \quad \text{for } 1 \leqslant i \leqslant n \ \ ,$$

where $x_1, \ldots, x_n$ are given covariates. Discuss carefully how to estimate β, and how to test that the model fits.

(ii) Carmichael *et al.* (1989) collected data on the numbers of 5-year old children with "dmft", i.e. with 5 or more decayed, missing or filled teeth, classified by social class, and by whether or not their tap water was fluoridated or non-fluoridated. The numbers of such children with dmft, and the total numbers, are given in the table below:

dmft

Social Class	Fluoridated	Non-fluoridated
I	12/117	12/56
II	26/170	48/146
III	11/52	29/64
Unclassified	24/118	49/104

A (slightly edited) version of the *R* output is given below. Explain carefully what model is being fitted, whether it does actually fit, and what the parameter estimates and Std. Errors are telling you. (You may assume that the factors SClass (social class) and Fl (with/without) have been correctly set up.)

```
Call:

     glm(formula = Yes/Total ~ SClass + Fl, family = binomial,
          weights = Total)

Coefficients:

                 Estimate  Std.  Error  z value
     (Intercept) -2.2716       0.2396   -9.480
     SClassII     0.5099       0.2628    1.940
     SClassIII    0.9857       0.3021    3.262
     SClassUnc    1.0020       0.2684    3.734
     Flwithout    1.0813       0.1694    6.383

(Dispersion parameter for binomial family taken to be 1)

     Null deviance:  68.53785 on 7 degrees of freedom

Residual deviance:  0.64225 on 3 degrees of freedom

Number of Fisher Scoring iterations:  3
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Here 'Yes' is the vector of numbers with dmft, taking values $12, 12, \ldots, 24, 49$, 'Total' is the vector of Total in each category, taking values $117, 56, \ldots, 118, 104$, and SClass, Fl are the factors corresponding to Social class and Fluoride status, defined in the obvious way.

15F Foundations of Quantum Mechanics

(i) The two states of a spin-$\frac{1}{2}$ particle corresponding to spin pointing along the z axis are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$. Explain why the states

$$|\uparrow,\theta\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle, \qquad |\downarrow,\theta\rangle = -\sin\frac{\theta}{2}\,|\uparrow\rangle + \cos\frac{\theta}{2}\,|\downarrow\rangle$$

correspond to the spins being aligned along a direction at an angle θ to the z direction.

The spin-0 state of two spin-$\frac{1}{2}$ particles is

$$|0\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\Big).$$

Show that this is independent of the direction chosen to define $|\uparrow\rangle_{1,2}$, $|\downarrow\rangle_{1,2}$. If the spin of particle 1 along some direction is measured to be $\frac{1}{2}\hbar$ show that the spin of particle 2 along the same direction is determined, giving its value.

[*The Pauli matrices are given by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad]$$

(ii) Starting from the commutation relation for angular momentum in the form

$$[J_3, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_3,$$

obtain the possible values of j, m, where $m\hbar$ are the eigenvalues of J_3 and $j(j+1)\hbar^2$ are the eigenvalues of $\mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$. Show that the corresponding normalized eigenvectors, $|j,m\rangle$, satisfy

$$J_\pm|j,m\rangle = \hbar\left((j\mp m)(j\pm m+1)\right)^{1/2}|j, m{\pm}1\rangle,$$

and that

$$\frac{1}{n!}J_-^{\,n}|j,j\rangle = \hbar^n\left(\frac{(2j)!}{n!(2j-n)!}\right)^{1/2}|j, j{-}n\rangle, \qquad n \leq 2j.$$

The state $|w\rangle$ is defined by

$$|w\rangle = e^{wJ_-/\hbar}|j,j\rangle,$$

for any complex w. By expanding the exponential show that $\langle w|w\rangle = (1+|w|^2)^{2j}$. Verify that

$$e^{-wJ_-/\hbar}J_3\,e^{wJ_-/\hbar} = J_3 - wJ_-,$$

and hence show that

$$J_3|w\rangle = \hbar\left(j - w\frac{\partial}{\partial w}\right)|w\rangle.$$

If $H = \alpha J_3$ verify that $|e^{i\alpha t}\rangle\, e^{-ij\alpha t}$ is a solution of the time-dependent Schrödinger equation.

16F Quantum Physics

A harmonic oscillator of frequency ω is in thermal equilibrium with a heat bath at temperature T. Show that the mean number of quanta n in the oscillator is

$$n = \frac{1}{e^{\hbar\omega/kT} - 1}.$$

Use this result to show that the density of photons of frequency ω for cavity radiation at temperature T is

$$n(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}.$$

By considering this system in thermal equilibrium with a set of distinguishable atoms, derive formulae for the Einstein A and B coefficients.

Give a brief description of the operation of a laser.

17J General Relativity

Discuss how Einstein's theory of gravitation reduces to Newton's in the limit of weak fields. Your answer should include discussion of:

(a) the field equations;

(b) the motion of a point particle;

(c) the motion of a pressureless fluid.

[*The metric in a weak gravitational field, with Newtonian potential ϕ, may be taken as*

$$ds^2 = dx^2 + dy^2 + dz^2 - (1 + 2\phi)dt^2.$$

The Riemann tensor is

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc}.$$]

18J Statistical Physics and Cosmology

(i) Given that $g(p)dp$ is the number of eigenstates of a gas particle with momentum between p and $p + dp$, write down the Bose-Einstein distribution $\bar{n}(p)$ for the average number of particles with momentum between p and $p + dp$, as a function of temperature T and chemical potential μ.

Given that $\mu = 0$ and $g(p) = 8\pi \frac{Vp^2}{h^3}$ for a gas of photons, obtain a formula for the energy density ρ_T at temperature T in the form

$$\rho_T = \int_0^\infty \epsilon_T(\nu) d\nu,$$

where $\epsilon_T(\nu)$ is a function of the photon frequency ν that you should determine. Hence show that the value ν_{peak} of ν at the maximum of $\epsilon_T(\nu)$ is proportional to T.

A thermally isolated photon gas undergoes a slow change of its volume V. Why is $\bar{n}(p)$ unaffected by this change? Use this fact to show that VT^3 remains constant.

(ii) According to the "Hot Big Bang" theory, the Universe evolved by expansion from an earlier state in which it was filled with a gas of electrons, protons and photons (with $n_e = n_p$) at thermal equilibrium at a temperature T such that

$$2m_e c^2 \gg kT \gg B \quad ,$$

where m_e is the electron mass and B is the binding energy of a hydrogen atom. Why must the composition have been different when $kT \gg 2m_e c^2$? Why must it change as the temperature falls to $kT \ll B$? Why does this lead to a thermal decoupling of radiation from matter?

The baryon number of the Universe can be taken to be the number of protons, either as free particles or as hydrogen atom nuclei. Let n_b be the baryon number density and n_γ the photon number density. Why is the ratio $\eta = n_b/n_\gamma$ unchanged by the expansion of the universe? Given that $\eta \ll 1$, obtain an estimate of the temperature T_D at which decoupling occurs, as a function of η and B. How does this decoupling lead to the concept of a "surface of last scattering" and a prediction of a Cosmic Microwave Background Radiation (CMBR)?

19H Transport Processes

Fluid flows in the x-direction past the infinite plane $y = 0$ with uniform but time-dependent velocity $U(t) = U_0 G(t/t_0)$, where G is a positive function with timescale t_0. A long region of the plane, $0 < x < L$, is heated and has temperature $T_0 (1 + \gamma (x/L)^n)$, where $T_0,\ \gamma,\ n$ are constants $[\gamma = O(1)]$; the remainder of the plane is insulating ($T_y = 0$). The fluid temperature far from the heated region is T_0. A thermal boundary layer is formed over the heated region. The full advection–diffusion equation for temperature $T(x, y, t)$ is

$$T_t + U(t)T_x = D\left(T_{yy} + T_{xx}\right), \qquad (1)$$

where D is the thermal diffusivity. By considering the steady case ($G \equiv 1$), derive a scale for the thickness of the boundary layer, and explain why the term T_{xx} in (1) can be neglected if $U_0 L/D \gg 1$.

Neglecting T_{xx}, use the change of variables

$$\tau = \frac{t}{t_0}, \quad \xi = \frac{x}{L}, \quad \eta = y\left[\frac{U(t)}{Dx}\right]^{1/2}, \quad \frac{T - T_0}{T_0} = \gamma\left(\frac{x}{L}\right)^n f(\xi, \eta, \tau)$$

to transform the governing equation to

$$f_{\eta\eta} + \frac{1}{2}\eta f_\eta - nf = \xi f_\xi + \frac{L\xi}{t_0 U_0}\left(\frac{G_\tau}{2G^2}\eta f_\eta + \frac{1}{G}f_\tau\right). \qquad (2)$$

Write down the boundary conditions to be satisfied by f in the region $0 < \xi < 1$.

In the case in which U is slowly-varying, so $\epsilon = \frac{L}{t_0 U_0} \ll 1$, consider a solution for f in the form

$$f = f_0(\eta) + \epsilon f_1(\xi, \eta, \tau) + O\left(\epsilon^2\right).$$

Explain why f_0 is independent of ξ and τ.

Henceforth take $n = \frac{1}{2}$. Calculate $f_0(\eta)$ and show that

$$f_1 = \frac{G_\tau \xi}{G^2} g(\eta) \ ,$$

where g satisfies the ordinary differential equation

$$g'' + \frac{1}{2}\eta g' - \frac{3}{2}g = \frac{-\eta}{4}\int_\eta^\infty e^{-u^2/4}du.$$

State the boundary conditions on $g(\eta)$.

The heat transfer per unit length of the heated region is $-DT_y|_{y=0}$. Use the above results to show that the total rate of heat transfer is

$$T_0\left[DLU(t)\right]^{1/2}\frac{\gamma}{2}\left\{\sqrt{\pi} - \frac{\epsilon G_\tau}{G^2}g'(0) + O\left(\epsilon^2\right)\right\}.$$

20L Theoretical Geophysics

Write down expressions for the phase speed c and group velocity c_g in one dimension for general waves of the form $A\exp[i(kx-\omega t)]$ with dispersion relation $\omega(k)$. Briefly indicate the physical significance of c and c_g for a wavetrain of finite size.

The dispersion relation for internal gravity waves with wavenumber $\mathbf{k}=(k,0,m)$ in an incompressible stratified fluid with constant buoyancy frequency N is

$$\omega = \frac{\pm Nk}{(k^2+m^2)^{1/2}}.$$

Calculate the group velocity $\mathbf{c}_g$ and show that it is perpendicular to $\mathbf{k}$. Show further that the horizontal components of $\mathbf{k}/\omega$ and $\mathbf{c}_g$ have the same sign and that the vertical components have the opposite sign.

The vertical velocity w of small-amplitude internal gravity waves is governed by

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right) + N^2\nabla_h^2 w = 0\ , \qquad (*)$$

where ∇_h^2 is the horizontal part of the Laplacian and N is constant.

Find separable solutions to $(*)$ of the form $w(x,z,t)=X(x-Ut)Z(z)$ corresponding to waves with constant horizontal phase speed $U>0$. Comment on the nature of these solutions for $0<k<N/U$ and for $k>N/U$.

A semi-infinite stratified fluid occupies the region $z>h(x,t)$ above a moving lower boundary $z=h(x,t)$. Construct the solution to $(*)$ for the case $h=\epsilon\sin[k(x-Ut)]$, where ϵ and k are constants and $\epsilon \ll 1$.

Sketch the orientation of the wavecrests, the propagation direction and the group velocity for the case $0<k<N/U$.

21H Mathematical Methods

The equation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} ,$$

where $\mathbf{A}$ is a real square matrix and $\mathbf{x}$ a column vector, has a simple eigenvalue $\lambda = \mu$ with corresponding right-eigenvector $\mathbf{x} = \mathbf{X}$. Show how to find expressions for the perturbed eigenvalue and right-eigenvector solutions of

$$\mathbf{A}\mathbf{x} + \epsilon \mathbf{b}(\mathbf{x}) = \lambda \mathbf{x} , \quad |\epsilon| \ll 1 ,$$

to first order in ϵ, where $\mathbf{b}$ is a vector function of $\mathbf{x}$. State clearly any assumptions you make.

If $\mathbf{A}$ is $(n \times n)$ and has a complete set of right-eigenvectors $\mathbf{X}^{(j)},\ j = 1, 2, ...n$, which span $\mathbb{R}^n$ and correspond to separate eigenvalues $\mu^{(j)},\ j = 1, 2, ...n$, find an expression for the first-order perturbation to $\mathbf{X}^{(1)}$ in terms of the $\left\{\mathbf{X}^{(j)}\right\}$ and the corresponding left-eigenvectors of $\mathbf{A}$.

Find the normalised eigenfunctions and eigenvalues of the equation

$$\frac{d^2y}{dx^2} + \lambda y = 0,\ 0 < x < 1 ,$$

with $y(0) = y(1) = 0$. Let these be the zeroth order approximations to the eigenfunctions of

$$\frac{d^2y}{dx^2} + \lambda y + \epsilon b(y) = 0,\ 0 < x < 1 ,$$

with $y(0) = y(1) = 0$ and where b is a function of y. Show that the first-order perturbations of the eigenvalues are given by

$$\lambda_n^{(1)} = -\epsilon\sqrt{2} \int_0^1 \sin(n\pi x) \;\; b\left(\sqrt{2}\sin n\pi x\right) dx .$$

22K Numerical Analysis

Write an essay on the computation of eigenvalues and eigenvectors of matrices.