UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 7 June 2001 9 to 12

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions in Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, ..., G*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1A Analysis II

State and prove the contraction mapping theorem.

Let $A = \{x, y, z\}$, let d be the discrete metric on A, and let d' be the metric given by: d' is symmetric and

$$d'(x, y) = 2,\ d'(x, z) = 2,\ d'(y, z) = 1,$$

$$d'(x, x) = d'(y, y) = d'(z, z) = 0.$$

Verify that d' is a metric, and that it is Lipschitz equivalent to d.

Define an appropriate function $f : A \to A$ such that f is a contraction in the d' metric, but not in the d metric.

2G Methods

Show that the symmetric and antisymmetric parts of a second-rank tensor are themselves tensors, and that the decomposition of a tensor into symmetric and antisymmetric parts is unique.

For the tensor A having components

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix},$$

find the scalar a, vector $\mathbf{p}$ and symmetric traceless tensor B such that

$$A\mathbf{x} = a\mathbf{x} + \mathbf{p} \wedge \mathbf{x} + B\mathbf{x}$$

for every vector $\mathbf{x}$.

3D Statistics

Suppose the **single** random variable X has a uniform distribution on the interval $[0, \theta]$ and it is required to estimate θ with the loss function

$$L(\theta, a) = c(\theta - a)^2,$$

where $c > 0$.

Find the posterior distribution for θ and the optimal Bayes point estimate with respect to the prior distribution with density $p(\theta) = \theta e^{-\theta},\ \theta > 0$.

4B Further Analysis

Define the terms *connected* and *path connected* for a topological space. If a topological space X is path connected, prove that it is connected.

Consider the following subsets of $\mathbb{R}^2$:

$$I = \{(x,0) : \ 0 \leq x \leq 1\}, \quad A = \{(0,y) : \ \tfrac{1}{2} \leq y \leq 1\}, \text{ and}$$

$$J_n = \{(n^{-1}, \ y) : \ 0 \leq y \leq 1\} \quad \text{for } n \geq 1.$$

Let

$$X = A \cup I \cup \bigcup_{n \geq 1} J_n$$

with the subspace (metric) topology. Prove that X is connected.

[*You may assume that any interval in* $\mathbb{R}$ *(with the usual topology) is connected.*]

5E Numerical Analysis

Find an LU factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 & 2 \\ -4 & 3 & -4 & -2 \\ 4 & -2 & 3 & 6 \\ -6 & 5 & -8 & 1 \end{pmatrix},$$

and use it to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 4 \\ 11 \end{pmatrix}.$$

6C Linear Mathematics

Show that right multiplication by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$ defines a linear transformation $\rho_A : M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$. Find the matrix representing ρ_A with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_{2\times 2}(\mathbb{C})$. Prove that the characteristic polynomial of ρ_A is equal to the square of the characteristic polynomial of A, and that A and ρ_A have the same minimal polynomial.

7E Complex Methods

A complex function is defined for every $z \in V$, where V is a non-empty open subset of $\mathbb{C}$, and it possesses a derivative at every $z \in V$. Commencing from a formal definition of derivative, deduce the Cauchy–Riemann equations.

8B Quadratic Mathematics

Let V be a finite-dimensional vector space over a field k. Describe a bijective correspondence between the set of bilinear forms on V, and the set of linear maps of V to its dual space V^*. If ϕ_1, ϕ_2 are non-degenerate bilinear forms on V, prove that there exists an isomorphism $\alpha : V \to V$ such that $\phi_2(u, v) = \phi_1(u, \alpha v)$ for all $u, v \in V$. If furthermore both ϕ_1, ϕ_2 are symmetric, show that α is self-adjoint (i.e. equals its adjoint) with respect to ϕ_1.

9F Quantum Mechanics

Consider a solution $\psi(x, t)$ of the time-dependent Schrödinger equation for a particle of mass m in a potential $V(x)$. The expectation value of an operator $\mathcal{O}$ is defined as

$$\langle\, \mathcal{O}\, \rangle = \int dx\ \psi^*(x,t)\ \mathcal{O}\ \psi(x,t).$$

Show that

$$\frac{d}{dt}\langle x \rangle = \frac{\langle p \rangle}{m},$$

where

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x},$$

and that

$$\frac{d}{dt}\langle p \rangle = \left\langle -\frac{\partial V}{\partial x}(x) \right\rangle.$$

[*You may assume that $\psi(x, t)$ vanishes as $x \to \pm\infty$.*]

SECTION II

10A Analysis II

Define total boundedness for metric spaces.

Prove that a metric space has the Bolzano–Weierstrass property if and only if it is complete and totally bounded.

11G Methods

Explain what is meant by an *isotropic* tensor.

Show that the fourth-rank tensor

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \qquad (*)$$

is isotropic for arbitrary scalars α, β and γ.

Assuming that the most general isotropic tensor of rank 4 has the form $(*)$, or otherwise, evaluate

$$B_{ijkl} = \int_{r<a} x_i x_j \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV,$$

where $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$.

12D Statistics

What is meant by a *generalized likelihood ratio test*? Explain in detail how to perform such a test.

Let $X_1, \ldots, X_n$ be independent random variables, and let X_i have a Poisson distribution with unknown mean $\lambda_i,\ i = 1, \ldots, n$.

Find the form of the generalized likelihood ratio statistic for testing $H_0 : \lambda_1 = \ldots = \lambda_n$, and show that it may be approximated by

$$\frac{1}{\bar{X}} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

If, for $n = 7$, you found that the value of this statistic was 27.3, would you accept H_0? Justify your answer.

13A Further Analysis

State Liouville's Theorem. Prove it by considering

$$\int_{|z|=R} \frac{f(z)\, dz}{(z-a)(z-b)}$$

and letting $R \to \infty$.

Prove that, if $g(z)$ is a function analytic on all of $\mathbb{C}$ with real and imaginary parts $u(z)$ and $v(z)$, then either of the conditions:

$$\text{(i) } u + v \geqslant 0 \text{ for all } z; \quad \text{or} \quad \text{(ii) } uv \geqslant 0 \text{ for all } z,$$

implies that $g(z)$ is constant.

14E Numerical Analysis

(a) Let B be an $n \times n$ positive-definite, symmetric matrix. Define the Cholesky factorization of B and prove that it is unique.

(b) Let A be an $m \times n$ matrix, $m \geqslant n$, such that $\text{rank} A = n$. Prove the uniqueness of the "skinny QR factorization"

$$A = QR,$$

where the matrix Q is $m \times n$ with orthonormal columns, while R is an $n \times n$ upper-triangular matrix with positive diagonal elements.

[*Hint: Show that you may choose R as a matrix that features in the Cholesky factorization of $B = A^T A$.*]

15C Linear Mathematics

Define the dual V^* of a vector space V. Given a basis $\{v_1, \ldots, v_n\}$ of V define its dual and show it is a basis of V^*. For a linear transformation $\alpha : V \to W$ define the dual $\alpha^* : W^* \to V^*$.

Explain (with proof) how the matrix representing $\alpha : V \to W$ with respect to given bases of V and W relates to the matrix representing $\alpha^* : W^* \to V^*$ with respect to the corresponding dual bases of V^* and W^*.

Prove that α and α^* have the same rank.

Suppose that α is an invertible endomorphism. Prove that $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

16E Complex Methods

Let R be a rational function such that $\lim_{z\to\infty}\{zR(z)\} = 0$. Assuming that R has no real poles, use the residue calculus to evaluate

$$\int_{-\infty}^{\infty} R(x)dx.$$

Given that $n \geqslant 1$ is an integer, evaluate

$$\int_0^{\infty} \frac{dx}{1+x^{2n}}.$$

17B Quadratic Mathematics

Suppose p is an odd prime and a an integer coprime to p. Define the Legendre symbol $(\frac{a}{p})$, and state (without proof) Euler's criterion for its calculation.

For j any positive integer, we denote by r_j the (unique) integer with $|r_j| \leq (p-1)/2$ and $r_j \equiv aj \bmod p$. Let l be the number of integers $1 \leq j \leq (p-1)/2$ for which r_j is negative. Prove that

$$\left(\frac{a}{p}\right) = (-1)^l.$$

Hence determine the odd primes for which 2 is a quadratic residue.

Suppose that $p_1, \ldots, p_m$ are primes congruent to 7 modulo 8, and let

$$N \;=\; 8(p_1 \ldots p_m)^2 - 1.$$

Show that 2 is a quadratic residue for any prime dividing N. Prove that N is divisible by some prime $p \equiv 7 \bmod 8$. Hence deduce that there are infinitely many primes congruent to 7 modulo 8.

18F Quantum Mechanics

(a) Write down the angular momentum operators L_1, L_2, L_3 in terms of x_i and

$$p_i = -i\hbar\frac{\partial}{\partial x_i},\; i = 1, 2, 3.$$

Verify the commutation relation

$$[L_1, L_2] = i\hbar L_3.$$

Show that this result and its cyclic permutations imply

$$[L_3, L_1 \pm iL_2] = \pm\hbar\,(L_1 \pm iL_2),$$
$$[\mathbf{L}^2, L_1 \pm iL_2] = 0.$$

(b) Consider a wavefunction of the form $\psi = (x_3^2 + ar^2)f(r)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. Show that for a particular value of a, ψ is an eigenfunction of both $\mathbf{L}^2$ and L_3. What are the corresponding eigenvalues?

END OF PAPER