UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 4 June 2001 9.00 to 12.00

PAPER 4

Before you begin read these instructions carefully.

*Each question in Section II carries twice the credit of each question in Section I. You may attempt **all four** questions in Section I and at most **five** questions from Section II. In Section II no more than **three** questions on each course may be attempted.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in two bundles, marked **A** and **E** according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet <u>must</u> bear your examination number and desk number.

SECTION I

1E Numbers and Sets

(a) Show that, given a set X, there is no bijection between X and its power set.

(b) Does there exist a set whose members are precisely those sets that are not members of themselves? Justify your answer.

2E Numbers and Sets

Prove, by induction or otherwise, that

$$\binom{n}{0}+\binom{n+1}{1}+\cdots+\binom{n+m}{m}=\binom{n+m+1}{m}.$$

Find the number of sequences consisting of zeroes and ones that contain exactly n zeroes and at most m ones.

3A Dynamics

Derive the equation

$$\frac{d^2u}{d\theta^2}+u=\frac{f(u)}{mh^2u^2}$$

for the motion of a particle of mass m under an attractive central force f, where $u=1/r$ and r is the distance of the particle from the centre of force, and where mh is the angular momentum of the particle about the centre of force.

[*Hint: you may assume the expressions for the radial and transverse accelerations in the form* $\ddot{r}-r\dot{\theta}^2, 2\dot{r}\dot{\theta}+r\ddot{\theta}$.]

4A Dynamics

Two particles of masses m_1 and m_2 at positions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are subject to forces $\mathbf{F}_1=-\mathbf{F}_2=\mathbf{f}(\mathbf{x}_1-\mathbf{x}_2)$. Show that the centre of mass moves at a constant velocity. Obtain the equation of motion for the relative position of the particles. How does the reduced mass

$$\mu=\frac{m_1m_2}{m_1+m_2}$$

of the system enter?

SECTION II

5E Numbers and Sets

(a) Prove Wilson's theorem, that $(p-1)! \equiv -1 \pmod{p}$, where p is prime.

(b) Suppose that p is an odd prime. Express $1^2.3^2.5^2.\ldots.(p-2)^2 \pmod{p}$ as a power of -1.

[*Hint:* $k \equiv -(p-k) \pmod{p}$.]

6E Numbers and Sets

State and prove the principle of inclusion-exclusion. Use it to calculate $\phi(4199)$, where ϕ is Euler's ϕ-function.

In a certain large college, a survey revealed that 90% of the fellows detest at least one of the pop stars Hairy, Dirty and Screamer. 45% detest Hairy, 28% detest Dirty and 46% detest Screamer. If 27% detest only Screamer and 6% detest all three, what proportion detest Hairy and Dirty but not Screamer?

7E Numbers and Sets

(a) Prove that, if p is prime and a is not a multiple of p, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) The *order* of $a \pmod{p}$ is the least positive integer d such that $a^d \equiv 1 \pmod{p}$. Suppose now that $a^x \equiv 1 \pmod{p}$; what can you say about x in terms of d? Show that $p \equiv 1 \pmod{d}$.

(c) Suppose that p is an odd prime. What is the order of $x \pmod{p}$ if $x^2 \equiv -1 \pmod{p}$? Find a condition on $p \pmod{4}$ that is equivalent to the existence of an integer x with $x^2 \equiv -1 \pmod{p}$.

8E Numbers and Sets

What is the Principle of Mathematical Induction? Derive it from the statement that every non-empty set of positive integers has a least element.

Prove, by induction on n, that $9^n \equiv 2^n \pmod 7$ for all $n \geq 1$.

What is wrong with the following argument?

"Theorem: $\sum_{i=1}^{n} i = n(n+1)/2 + 126$.

Proof: Assume that $m \geq 1$ and $\sum_{i=1}^{m} i = m(m+1)/2 + 126$. Add $m+1$ to both sides to get

$$\sum_{i=1}^{m+1} i \;=\; m(m+1)/2 + m + 1 + 126 \;=\; (m+1)(m+2)/2 + 126.$$

So, by induction, the theorem is proved."

9A Dynamics

The position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. Write down the equation of motion of the particle under a force $\mathbf{F} = -4m\omega^2\mathbf{x}$.

Find the motion of the particle in (x, y, z) coordinates with initial condition

$$\mathbf{x} = (1,0,0) \quad \text{and} \quad \dot{\mathbf{x}} = (0,0,0) \qquad \text{at } t = 0,$$

where $\boldsymbol{\omega} = (0, 0, \omega)$. Show that the particle has a maximum speed at $t = (2n+1)\pi/4\omega$, and find this speed.

[*Hint: you may find it useful to consider the combination* $\zeta = x + iy$.]

10A Dynamics

A spherical raindrop of radius $a(t) > 0$ and density ρ falls down at a velocity $v(t) > 0$ through a fine stationary mist. As the raindrop falls its volume grows at the rate $c\pi a^2 v$ with constant c. The raindrop is subject to the gravitational force and a resistive force $-k\rho\pi a^2 v^2$ with k a positive constant. Show a and v satisfy

$$\dot{a} = \tfrac{1}{4}cv,$$
$$\dot{v} = g - \tfrac{3}{4}(c+k)\frac{v^2}{a}.$$

Find an expression for $\frac{d}{dt}(v^2/a)$, and deduce that as time increases v^2/a tends to the constant value $g/(\frac{7}{8}c + \frac{3}{4}k)$, and thence the raindrop tends to a constant acceleration which is less than $\frac{1}{7}g$.

11A Dynamics

A spacecraft of mass m moves under the gravitational influence of the Sun of mass M and with universal gravitation constant G. After a disastrous manoeuvre, the unfortunate spacecraft finds itself exactly in a parabolic orbit about the Sun: the orbit with zero total energy. Using the conservation of energy and angular momentum, or otherwise, show that in the subsequent motion the distance of the spacecraft from the Sun $r(t)$ satisfies

$$(r - r_0)(r + 2r_0)^2 = \tfrac{9}{2} GM (t - t_0)^2,$$

with constants r_0 and t_0.

12A Dynamics

Find the moment of inertia of a uniform solid cylinder of radius a, length l and total mass M about its axis.

The cylinder is released from rest at the top of an inclined plane of length L and inclination θ to the horizontal. The first time the plane is perfectly smooth and the cylinder slips down the plane without rotating. The experiment is then repeated after the plane has been roughened, so that the cylinder now rolls without slipping at the point of contact. Show that the time taken to roll down the roughened plane is $\sqrt{\frac{3}{2}}$ times the time taken to slip down the smooth plane.

END OF PAPER