UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 1 June 2001 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I and at most* ***five*** *questions from Section II. In Section II no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in two bundles, marked ***B*** *and* ***F*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet *must* ***bear your examination number and desk number.***

SECTION I

1B Differential Equations

Find the solution to

$$\frac{dy(x)}{dx} + \tanh(x)\, y(x) \;=\; H(x) \;,$$

in the range $-\infty < x < \infty$ subject to $y(0) = 1$, where $H(x)$ is the Heavyside function defined by

$$H(x) \;=\; \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \;.$$

Sketch the solution.

2B Differential Equations

The function $y(x)$ satisfies the inhomogeneous second-order linear differential equation

$$y'' - y' - 2y \;=\; 18xe^{-x} \;.$$

Find the solution that satisfies the conditions that $y(0) = 1$ and $y(x)$ is bounded as $x \to \infty$.

3F Probability

The following problem is known as Bertrand's paradox. A chord has been chosen at random in a circle of radius r. Find the probability that it is longer than the side of the equilateral triangle inscribed in the circle. Consider three different cases:

a) the middle point of the chord is distributed uniformly inside the circle,

b) the two endpoints of the chord are independent and uniformly distributed over the circumference,

c) the distance between the middle point of the chord and the centre of the circle is uniformly distributed over the interval $[0, r]$.

[*Hint: drawing diagrams may help considerably.*]

4F Probability

The Ruritanian authorities decided to pardon and release one out of three remaining inmates, A, B and C, kept in strict isolation in the notorious Alkazaf prison. The inmates know this, but can't guess who among them is the lucky one; the waiting is agonising. A sympathetic, but corrupted, prison guard approaches A and offers to name, in exchange for a fee, another inmate (not A) who is doomed to stay. He says: "This reduces your chances to remain here from 2/3 to 1/2: will it make you feel better?" A hesitates but then accepts the offer; the guard names B.

Assume that indeed B will not be released. Determine the conditional probability

$$P\big(A \text{ remains } \bigm| B \text{ named}\big) \;=\; \tfrac{P(A \& B \text{ remain})}{P(B \text{ named})}$$

and thus check the guard's claim, in three cases:

a) when the guard is completely unbiased (i.e., names any of B and C with probability 1/2 if the pair B, C is to remain jailed),

b) if he hates B and would certainly name him if B is to remain jailed,

c) if he hates C and would certainly name him if C is to remain jailed.

SECTION II

5B Differential Equations

The real sequence $y_k,\ k = 1, 2, \ldots$ satisfies the difference equation

$$y_{k+2} - y_{k+1} + y_k \;=\; 0 \;.$$

Show that the general solution can be written

$$y_k \;=\; a\cos\frac{\pi k}{3} \;+\; b\sin\frac{\pi k}{3}\;,$$

where a and b are arbitrary real constants.

Now let y_k satisfy

$$y_{k+2} - y_{k+1} + y_k \;=\; \frac{1}{k+2}\;. \qquad (*)$$

Show that a particular solution of $(*)$ can be written in the form

$$y_k \;=\; \sum_{n=1}^{k} \frac{a_n}{k-n+1}\;,$$

where

$$a_{n+2} - a_{n+1} + a_n = 0\;, \qquad n \geq 1\;,$$

and $a_1 = 1,\ a_2 = 1\;.$

Hence, find the general solution to $(*)$.

6B Differential Equations

The function $y(x)$ satisfies the linear equation

$$y''(x) + p(x)y'(x) + q(x)y(x) \;=\; 0\ .$$

The Wronskian, $W(x)$, of two independent solutions denoted $y_1(x)$ and $y_2(x)$ is defined to be

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix}\ .$$

Let $y_1(x)$ be given. In this case, show that the expression for $W(x)$ can be interpreted as a first-order inhomogeneous differential equation for $y_2(x)$. Hence, by explicit derivation, show that $y_2(x)$ may be expressed as

$$y_2(x) \;=\; y_1(x) \int_{x_0}^{x} \frac{W(t)}{y_1(t)^2}\, dt\ , \qquad (*)$$

where the rôle of x_0 should be briefly elucidated.

Show that $W(x)$ satisfies

$$\frac{dW(x)}{dx} + p(x)W(x) \;=\; 0\ .$$

Verify that $y_1(x) = 1 - x$ is a solution of

$$xy''(x) - (1-x^2)y'(x) - (1+x)y(x) \;=\; 0\ . \qquad (\dagger)$$

Hence, using $(*)$ with $x_0 = 0$ and expanding the integrand in powers of t to order t^3, find the first three non-zero terms in the power series expansion for a solution, $y_2(x)$, of $(\dagger)$ that is independent of $y_1(x)$ and satisfies $y_2(0) = 0$, ${y_2}''(0) = 1$.

7B Differential Equations

Consider the linear system

$$\dot{\mathbf{z}} + A\mathbf{z} \;=\; \mathbf{h}\,, \qquad (*)$$

where

$$\mathbf{z}(t) \;=\; \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\,, \quad A \;=\; \begin{pmatrix} 1+a & -2 \\ 1 & -1+a \end{pmatrix}\,, \quad \mathbf{h}(t) \;=\; \begin{pmatrix} 2\cos t \\ \cos t - \sin t \end{pmatrix}\,,$$

where $\mathbf{z}(t)$ is real and a is a real constant, $a \geq 0$.

Find a (complex) eigenvector, $\mathbf{e}$, of A and its corresponding (complex) eigenvalue, l. Show that the second eigenvector and corresponding eigenvalue are respectively $\bar{\mathbf{e}}$ and $\bar{l}$, where the bar over the symbols signifies complex conjugation. Hence explain how the general solution to $(*)$ can be written as

$$\mathbf{z}(t) \;=\; \alpha(t)\,\mathbf{e} \;+\; \bar{\alpha}(t)\,\bar{\mathbf{e}}\,,$$

where $\alpha(t)$ is complex.

Write down a differential equation for $\alpha(t)$ and hence, for $a > 0$, deduce the solution to $(*)$ which satisfies the initial condition $\mathbf{z}(0) = \underline{0}$.

Is the linear system resonant?

By taking the limit $a \to 0$ of the solution already found deduce the solution satisfying $\mathbf{z}(0) = \underline{0}$ when $a = 0$.

8B Differential Equations

Carnivorous hunters of population h prey on vegetarians of population p. In the absence of hunters the prey will increase in number until their population is limited by the availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$\begin{aligned} \dot{p} \;&=\; p\left(1 - \frac{p}{a}\right) - \frac{ph}{a}\,, \\ \dot{h} \;&=\; \frac{h}{8}\left(\frac{p}{b} - 1\right)\,, \end{aligned} \qquad (*)$$

where a and b are positive constants, and $h(t)$ and $p(t)$ are non-negative functions of time, t. By giving an interpretation of each term explain briefly how these equations model the system described.

Consider these equations for $a = 1$. In the two cases $0 < b < 1/2$ and $b > 1$ determine the location and the stability properties of the critical points of $(*)$. In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

9F Probability

I play tennis with my parents; the chances for me to win a game against Mum (M) are p and against Dad (D) q, where $0 < q < p < 1$. We agreed to have three games, and their order can be DMD (where I play against Dad, then Mum then again Dad) or MDM. The results of games are independent.

Calculate under each of the two orders the probabilities of the following events:

a) that I win at least one game,

b) that I win at least two games,

c) that I win at least two games in succession (i.e., games 1 and 2 or 2 and 3, or 1, 2 and 3),

d) that I win exactly two games in succession (i.e., games 1 and 2 or 2 and 3, but not 1, 2 and 3),

e) that I win exactly two games (i.e., 1 and 2 or 2 and 3 or 1 and 3, but not 1, 2 and 3).

In each case a)– e) determine which order of games maximizes the probability of the event. In case e) assume in addition that $p + q > 3pq$.

10F Probability

A random point is distributed uniformly in a unit circle $\mathcal{D}$ so that the probability that it falls within a subset $\mathcal{A} \subseteq \mathcal{D}$ is proportional to the area of $\mathcal{A}$. Let R denote the distance between the point and the centre of the circle. Find the distribution function $F_R(x) = P(R < x)$, the expected value ER and the variance $\text{Var } R = ER^2 - (ER)^2$.

Let Θ be the angle formed by the radius through the random point and the horizontal line. Prove that R and Θ are independent random variables.

Consider a coordinate system where the origin is placed at the centre of $\mathcal{D}$. Let X and Y denote the horizontal and vertical coordinates of the random point. Find the covariance $\text{Cov}(X, Y) = E(XY) - EXEY$ and determine whether X and Y are independent.

Calculate the sum of expected values $E\frac{X}{R} + iE\frac{Y}{R}$. Show that it can be written as the expected value $Ee^{i\xi}$ and determine the random variable ξ.

11F Probability

Dipkomsky, a desperado in the wild West, is surrounded by an enemy gang and fighting tooth and nail for his survival. He has m guns, $m > 1$, pointing in different directions and tries to use them in succession to give an impression that there are several defenders. When he turns to a subsequent gun and discovers that the gun is loaded he fires it with probability $1/2$ and moves to the next one. Otherwise, i.e. when the gun is unloaded, he loads it with probability $3/4$ or simply moves to the next gun with complementary probability $1/4$. If he decides to load the gun he then fires it or not with probability $1/2$ and after that moves to the next gun anyway.

Initially, each gun had been loaded independently with probability p. Show that if after each move this distribution is preserved, then $p = 3/7$. Calculate the expected value EN and variance Var N of the number N of loaded guns under this distribution.

[*Hint: it may be helpful to represent* N *as a sum* $\sum_{1 \leq j \leq m} X_j$ *of random variables taking values* 0 *and* 1.]

12F Probability

A taxi travels between four villages, W, X, Y, Z, situated at the corners of a rectangle. The four roads connecting the villages follow the sides of the rectangle; the distance from W to X and Y to Z is 5 miles and from W to Z and Y to X 10 miles. After delivering a customer the taxi waits until the next call then goes to pick up the new customer and takes him to his destination. The calls may come from any of the villages with probability $1/4$ and each customer goes to any other village with probability $1/3$. Naturally, when travelling between a pair of adjacent corners of the rectangle, the taxi takes the straight route, otherwise (when it travels from W to Y or X to Z or vice versa) it does not matter. Distances within a given village are negligible. Let D be the distance travelled to pick up and deliver a single customer. Find the probabilitites that D takes each of its possible values. Find the expected value ED and the variance Var D.

END OF PAPER