UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 31 May 2001 9.00 to 12.00

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I and at most* ***five*** *questions from Section II. In Section II no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in two bundles, marked ***C*** *and* ***D*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet <u>*must*</u> ***bear your examination number and desk number.***

SECTION I

1C Algebra and Geometry

Show, using the summation convention or otherwise, that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$, for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

The function $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\Pi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n})$ where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. Show that Π is linear and find the elements of a matrix P such that $\Pi(\mathbf{x}) = P\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Find all solutions to the equation $\Pi(\mathbf{x}) = \mathbf{x}$. Evaluate $\Pi(\mathbf{n})$. Describe the function Π geometrically. Justify your answer.

2C Algebra and Geometry

Define what is meant by the statement that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ are linearly independent. Determine whether the following vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ are linearly independent and justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

For the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ taken from a real vector space V consider the statements

A) $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent,
B) $\exists\, \alpha, \beta, \gamma \in \mathbb{R} : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
C) $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$, not all $= 0 : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
D) $\exists\, \alpha, \beta \in \mathbb{R}$, not both $= 0 : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
E) $\exists\, \alpha, \beta \in \mathbb{R} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
F) $\nexists$ basis of V that contains all 3 vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

State if the following implications are true or false (no justification is required):

i)	A $\Rightarrow$ B,	vi)	B $\Rightarrow$ A,
ii)	A $\Rightarrow$ C,	vii)	C $\Rightarrow$ A,
iii)	A $\Rightarrow$ D,	viii)	D $\Rightarrow$ A,
iv)	A $\Rightarrow$ E,	ix)	E $\Rightarrow$ A,
v)	A $\Rightarrow$ F,	x)	F $\Rightarrow$ A.

3D Analysis I

What does it mean to say that $u_n \to l$ as $n \to \infty$?

Show that, if $u_n \to l$ and $v_n \to k$, then $u_n v_n \to lk$ as $n \to \infty$.

If further $u_n \neq 0$ for all n and $l \neq 0$, show that $1/u_n \to 1/l$ as $n \to \infty$.

Give an example to show that the non-vanishing of u_n for all n need not imply the non-vanishing of l.

4D Analysis I

Starting from the theorem that any continuous function on a closed and bounded interval attains a maximum value, prove Rolle's Theorem. Deduce the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(t) > 0$ for all t show that f is a strictly increasing function.

Conversely, if f is strictly increasing, is $f'(t) > 0$ for all t?

SECTION II

5C Algebra and Geometry

The matrix

$$A_\alpha = \begin{pmatrix} 1 & -1 & 2\alpha+1 \\ 1 & \alpha-1 & 1 \\ 1+\alpha & -1 & \alpha^2+4\alpha+1 \end{pmatrix}$$

defines a linear map $\Phi_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_\alpha(\mathbf{x}) = A_\alpha \mathbf{x}$. Find a basis for the kernel of Φ_α for all values of $\alpha \in \mathbb{R}$.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be bases of $\mathbb{R}^3$. Show that there exists a matrix S, to be determined in terms of $\mathcal{B}$ and $\mathcal{C}$, such that, for every linear mapping Φ, if Φ has matrix A with respect to $\mathcal{B}$ and matrix A' with respect to $\mathcal{C}$, then $A' = S^{-1}AS$.

For the bases

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad , \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\} \quad ,$$

find the basis transformation matrix S and calculate $S^{-1}A_0S$.

6C Algebra and Geometry

Assume that $\mathbf{x}_p$ is a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x}, \mathbf{b} \in \mathbb{R}^3$ and a real 3×3 matrix A. Explain why the general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = \mathbf{x}_p + \mathbf{h}$ where $\mathbf{h}$ is any vector such that $A\mathbf{h} = \mathbf{0}$.

Now assume that A is a real symmetric 3×3 matrix with three different eigenvalues λ_1, λ_2 and λ_3. Show that eigenvectors of A with respect to different eigenvalues are orthogonal. Let $\mathbf{x}_k$ be a normalised eigenvector of A with respect to the eigenvalue λ_k, $k = 1, 2, 3$. Show that the linear system

$$(A - \lambda_k I)\mathbf{x} \; = \; \mathbf{b} \;\; ,$$

where I denotes the 3×3 unit matrix, is solvable if and only if $\mathbf{x}_k.\mathbf{b} = 0$. Show that the general solution is given by

$$\mathbf{x} \; = \; \sum_{i \neq k} \tfrac{\mathbf{b}.\mathbf{x}_i}{\lambda_i - \lambda_k} \mathbf{x}_i + \beta \mathbf{x}_k, \quad \beta \in \mathbb{R}.$$

[*Hint: consider the components of* $\mathbf{x}$ *and* $\mathbf{b}$ *with respect to a basis of eigenvectors of* A.]

Consider the matrix A and the vector $\mathbf{b}$

$$A = \begin{pmatrix} -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \sqrt{2} + \sqrt{3} \\ -\sqrt{2} + \sqrt{3} \\ -2\sqrt{3} \end{pmatrix}.$$

Verify that $\frac{1}{\sqrt{3}}(1,1,1)^T$ and $\frac{1}{\sqrt{2}}(1,-1,0)^T$ are eigenvectors of A. Show that $A\mathbf{x} = \mathbf{b}$ is solvable and find its general solution.

7C Algebra and Geometry

For $\alpha, \gamma \in \mathbb{R}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$ and $\beta\overline{\beta} \geqslant \alpha\gamma$ the equation $\alpha z\overline{z} - \beta\overline{z} - \overline{\beta}z + \gamma = 0$ describes a circle $C_{\alpha\beta\gamma}$ in the complex plane. Find its centre and radius. What does the equation describe if $\beta\overline{\beta} < \alpha\gamma$? Sketch the circles $C_{\alpha\beta\gamma}$ for $\beta = \gamma = 1$ and $\alpha = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1$.

Show that the complex function $f(z) = \beta\overline{z}/\overline{\beta}$ for $\beta \neq 0$ satisfies $f(C_{\alpha\beta\gamma}) = C_{\alpha\beta\gamma}$.

[*Hint:* $f(C) = C$ *means that* $f(z) \in C \;\forall z \in C$ *and* $\forall w \in C \;\; \exists z \in C$ *such that* $f(z) = w$.]

For two circles C_1 and C_2 a function $m(C_1, C_2)$ is defined by

$$m(C_1, C_2) \; = \; \max_{z \in C_1, w \in C_2} \left| z - w \right| \;\; .$$

Prove that $m(C_1, C_2) \leqslant m(C_1, C_3) + m(C_2, C_3)$. Show that

$$m(C_{\alpha_1\beta_1\gamma_1}, C_{\alpha_2\beta_2\gamma_2}) \; = \; \tfrac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{|\alpha_1\alpha_2|} + \tfrac{\sqrt{\beta_1\overline{\beta_1} - \alpha_1\gamma_1}}{|\alpha_1|} + \tfrac{\sqrt{\beta_2\overline{\beta_2} - \alpha_2\gamma_2}}{|\alpha_2|} \;\; .$$

8C Algebra and Geometry

Let $l_{\mathbf{x}}$ denote the straight line through $\mathbf{x}$ with directional vector $\mathbf{u} \neq \mathbf{0}$

$$l_{\mathbf{x}} = \left\{ \mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = \mathbf{x} + \lambda \mathbf{u}, \lambda \in \mathbb{R} \right\} \quad .$$

Show that $l_{\mathbf{0}}$ is a subspace of $\mathbb{R}^3$ and show that $l_{\mathbf{x_1}} = l_{\mathbf{x_2}} \Leftrightarrow \mathbf{x_1} = \mathbf{x}_2 + \lambda \mathbf{u}$ for some $\lambda \in \mathbb{R}$.

For fixed $\mathbf{u} \neq \mathbf{0}$ let $\mathcal{L}$ be the set of all the parallel straight lines $l_{\mathbf{x}}$ ($\mathbf{x} \in \mathbb{R}^3$) with directional vector $\mathbf{u}$. On $\mathcal{L}$ an addition and a scalar multiplication are defined by

$$l_{\mathbf{x}} + l_{\mathbf{y}} = l_{\mathbf{x}+\mathbf{y}}, \ \alpha l_{\mathbf{x}} = l_{\alpha \mathbf{x}}, \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \ \alpha \in \mathbb{R} \ .$$

Explain why these operations are well-defined. Show that the addition is associative and that there exists a zero vector which should be identified.

You may now assume that $\mathcal{L}$ is a vector space. If $\{\mathbf{u}, \mathbf{b}_1, \mathbf{b_2}\}$ is a basis for $\mathbb{R}^3$ show that $\{l_{\mathbf{b}_1}, l_{\mathbf{b}_2}\}$ is a basis for $\mathcal{L}$.

For $\mathbf{u} = (1, 3, -1)^T$ a linear map $\Phi : \mathcal{L} \to \mathcal{L}$ is defined by

$$\Phi\left(l_{(1,-1,0)^T}\right) = l_{(2,4,-1)^T} \ , \ \Phi\left(l_{(1,1,0)^T}\right) = l_{(-4,-2,1)^T} \quad .$$

Find the matrix A of Φ with respect to the basis $\{l_{(1,0,0)^T}, l_{(0,1,0)^T}\}$.

9D Analysis I

(i) If $a_0, a_1, \ldots$ are complex numbers show that if, for some $w \in \mathbb{C}, w \neq 0$, the set $\{|a_n w^n| : n \geq 0\}$ is bounded and $|z| < |w|$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Use this result to define the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$.

(ii) If $|a_n|^{1/n} \to R$ as $n \to \infty$ $(0 < R < \infty)$ show that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence equal to $1/R$.

(iii) Give examples of power series with radii of convergence 1 such that (a) the series converges at all points of the circle of convergence, (b) diverges at all points of the circle of convergence, and (c) neither of these occurs.

10D Analysis I

Suppose that f is a continuous real-valued function on $[a, b]$ with $f(a) < f(b)$. If $f(a) < v < f(b)$ show that there exists c with $a < c < b$ and $f(c) = v$.

Deduce that if f is a continuous function from the closed bounded interval $[a, b]$ to itself, there exists at least one fixed point, i.e., a number d belonging to $[a, b]$ with $f(d) = d$. Does this fixed point property remain true if f is a continuous function defined (i) on the open interval (a, b) and (ii) on $\mathbb{R}$? Justify your answers.

11D Analysis I

(i) Show that if $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable then, given $\epsilon > 0$, we can find some constant L and $\delta(\epsilon) > 0$ such that

$$|g(t) - g(\alpha) - g'(\alpha)\,(t - \alpha)| \;\leq\; L|t - \alpha|^2$$

for all $|t - \alpha| < \delta(\epsilon)$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable on $[a, b]$ (with one-sided derivatives at the end points), let f' and f'' be strictly positive functions and let $f(a) < 0 < f(b)$.

If $F(t) = t - (f(t)/f'(t))$ and a sequence $\{x_n\}$ is defined by $b = x_0, x_n = F(x_{n-1})$ $(n > 0)$, show that $x_0, x_1, x_2, \ldots$ is a decreasing sequence of points in $[a, b]$ and hence has limit α. What is $f(\alpha)$? Using part (i) or otherwise estimate the rate of convergence of x_n to α, i.e., the behaviour of the absolute value of $(x_n - \alpha)$ for large values of n.

12D Analysis I

Explain what it means for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$, and give an example of a bounded function that is not Riemann integrable.

Show each of the following statements is true for continuous functions f, but false for general Riemann integrable functions f.

(i) If $f : [a, b] \to \mathbb{R}$ is such that $f(t) \geq 0$ for all t in $[a, b]$ and $\int_a^b f(t)\,dt = 0$, then $f(t) = 0$ for all t in $[a, b]$.

(ii) $\int_a^t f(x)\,dx$ is differentiable and $\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$.

END OF PAPER