4 Dynamics

4.5 Euler's Equations (8 units)

This project is self-contained, though material from the Part II(C) course Classical Dynamics is relevant.

1 Introduction

The angular velocity with respect to principal axes of inertia in a rigid body is taken to be

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3). \tag{1}$$

If the principal moments of inertia are A, B, C with respect to these axes, then the angular momentum is

$$\mathbf{h} = (A\omega_1, B\omega_2, C\omega_3). \tag{2}$$

These axes are fixed in the body and have angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame instantaneously coincident with the principal axes. The rate of change of angular momentum with respect to such an inertial frame is

$$\frac{d\mathbf{h}}{dt} + \boldsymbol{\omega} \wedge \mathbf{h}. \tag{3}$$

In the case when there is no net moment of external forces acting on the body, the law "rate of change of angular momentum = moment of external forces" gives:

$$\frac{d\mathbf{h}}{dt} + \boldsymbol{\omega} \wedge \mathbf{h} \ = \mathbf{0}. \tag{4}$$

Expanding this equation into components gives:

$$\left.\begin{aligned} A\tfrac{d\omega_1}{dt} + (C-B)\omega_2\omega_3 = 0 \\ B\tfrac{d\omega_2}{dt} + (A-C)\omega_3\omega_1 = 0 \\ C\tfrac{d\omega_3}{dt} + (B-A)\omega_1\omega_2 = 0 \end{aligned}\right\} \tag{5}$$

It can be shown analytically that these equations have two first integrals, which say that the energy and the magnitude of the angular momentum remain constant, as follows:

$$\tfrac{1}{2}A\omega_1^2 + \tfrac{1}{2}B\omega_2^2 + \tfrac{1}{2}C\omega_3^2 = E \tag{6}$$

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2 \tag{7}$$

Since the moment of external forces is zero, we also know that the angular momentum vector $\mathbf{h}$ is constant when measured in an inertial frame.

2 Project work

2.1 Program requirements

Write a program to solve Euler's equations (5) numerically and plot the results. You should use MATLAB's 64-bit (8-byte) double-precision floating-point values or the equivalent in other programming languages. Output from your program should include:

1. Graphs of $\omega_1(t)$, $\omega_2(t)$ and $\omega_3(t)$ against t;
2. 3-D phase space plots of ω_1, ω_2 and ω_3. Choose the OX axis for ω_1, etc.

Equations (6) and (7) can be used to check the accuracy of your numerical results by calculating and displaying the values of these expressions at the beginning and end of runs.

The objective of the project is to investigate and classify all possible types of motion. The following questions provide some guidelines for the investigation.

Question 1 Since A, B, C, $\omega_1(0)$, $\omega_2(0)$ and $\omega_3(0)$ may all in principle take arbitrary values, the parameter space to be explored may seem very large. If A, B and C take distinct values, explain how the results from taking

$$A > B > C \tag{8}$$

can be generalised. Briefly explain what happens if any two (or all three) of A, B, C are equal. For given values of A/B and C/B, explain why we may take $B = 1$ without loss of generality.

Show further that choosing $E = 1$ is equivalent to re-scaling the time variable t, and give the scaling factor.

2.2 Results requirements

From this point on, take $B = 1$ and $E = 1$ and build these values into your program. Your program should allow you to set and change the values of A/B and C/B. Assume that $A/B > 1$ and $C/B < 1$, and work with $A = 1.4$, $B = 1$ and $C = 0.7$ unless other values are suggested. You may find it convenient to accept arbitrary input for $\omega_1(0)$, $\omega_2(0)$ and $\omega_3(0)$ and then scale the input values so that $E = 1$.

Question 2 Use your program to demonstrate that solutions are possible in which the vector $\boldsymbol{\omega}(t)$ rotates around the OX axis with small amplitude deviation from $(\sqrt{2/A}, 0, 0)$ (i.e., $(1, 0, 0)$ before scaling), and that similar stable solutions exist near the OZ axis. Include copies of your results.

Question 3 Approximate Euler's Equations (5) by linearising them for the cases where (i) $\omega_1 \approx \sqrt{2/A}$ and ω_2, ω_3 are small, and (ii) $\omega_3 \approx \sqrt{2/C}$ and ω_1, ω_2 are small. Describe the analytic solutions. Are your solutions consistent with the results obtained for question 2 above? Give analytic expressions for the period of the motion in each case and compare with your results. (You need not compute the periods; an estimate by eye from your graphs of ω vs t is sufficient.)

Question 4 Are your numerical results consistent with equations (6) and (7)? To what extent are further checks on numerical accuracy needed?

Question 5 What solutions do you obtain if starting conditions are chosen so that $\boldsymbol{\omega}(0)$ lies very close to the OY axis? Describe the motion physically.

Question 6 Make a plausible case based on your computed results that there exists a solution $\boldsymbol{\omega}(t)$ which begins away from the OY axis but which tends towards the steady (but unstable) solution parallel to the OY axis as $t \to \infty$. What happens if you attempt to simulate such a solution numerically? What value must H take for such a solution?

Question 7 Using the following scheme, classify all the possible qualitative types of motion of the system assuming A, B and C take distinct values. Take the solution discussed in question 6 as a type of its own, and use it to separate the remaining solutions into two types. Describe the range of behaviour observable for each type. Explain clearly how the solution discussed in question 6 divides the solution space into regions (can you find the equations of the boundaries of these regions?) and how solutions behave physically as the boundaries are approached.

Try different values of A/B and C/B. How does the choice of these parameters affect your results?

Question 8 The rigid body is now subjected to slow friction via a retarding couple $-k\boldsymbol{\omega}$, where k is a very small parameter. How does this affect equations (5), (6) and (7)? Alter your program to incorporate the couple and investigate a few types of solutions for the original values of A, B and C. You may find it useful to consider 3D phase plots of a *suitably normalised* $\boldsymbol{\omega}(t)$, as well as your normal plots.

Question 9 Is your classification in question 7 still of any use or has it become irrelevant? Is there still a division of the solution space into regions?

