3 Fluid and Solid Mechanics

3.9 Viscous Flow in a Collapsible Channel (9 units)

This project requires a knowledge of lubrication theory for a viscous fluid, as taught in the Part II course Fluid Dynamics II and described in [1].

1 Introduction

The tubes that carry fluid around the body (such as veins, arteries, lung airways, the urethra, etc.) have deformable walls. The shape of such a tube is strongly coupled to the flow within it through the internal pressure distribution. This nonlinear flow-structure interaction imparts to such systems unusual but biologically significant properties, notably "flow limitation" (so that airway flexibility limits the rate at which you can expel air from your lungs, for example). To explore such interactions, one can consider a simple model system in which an incompressible fluid flows steadily through a two-dimensional channel, one wall of which is formed by a membrane under longitudinal tension. Assuming that the channel is long and thin, and that the fluid is sufficiently viscous, lubrication theory can be used to describe the flow.

Suppose the channel lies in $0 \leqslant y \leqslant h(x)$, $0 \leqslant x \leqslant L$, where $L \gg h$. Applying no-slip and no-penetration conditions along the rigid wall $y = 0$ and the membrane $y = h$, the relationship between the steady, uniform flux q of fluid along the channel and the local pressure gradient p_x is approximately $q = -h^3 p_x/(12\mu)$, where μ is the fluid's viscosity, assumed constant. The fluid pressure distribution $p(x)$ is controlled by the shape of the channel wall according to $p = -Th_{xx}$, where T is the tension in the membrane, assumed constant; the pressure outside the membrane is taken to be zero. We assume that the membrane is fixed at either end, so that $h(0) = h(L) = h_0$, for some constant h_0. The flow is controlled by the upstream and downstream pressures $p(0) = p_u$ and $p(L) = p_d$, and characterised by the relationship between the flux q and the pressure drop along the channel, $p_u - p_d$, holding either p_u or p_d constant.

The problem can be simplified by nondimensionalisation. Let

$$h(x) = h_0 H(X), \quad x = LX, \quad \text{and} \quad p(x) = p_0 P(X),$$

where $p_0 = Th_0/L^2$. This yields nondimensional parameters $Q = 12\mu L^3 q/(Th_0^4)$, $P_u = p_u/p_0$, $P_d = p_d/p_0$ and governing equations

$$Q = -H^3 P_X, \qquad P = -H_{XX} \qquad (0 \leqslant X \leqslant 1) \tag{1}$$

subject to

$$H(0) = 1, \quad H(1) = 1, \quad P(0) = P_u, \quad P(1) = P_d. \tag{2}$$

We seek graphs of $\Delta P = P_u - P_d > 0$ as a function of Q, for fixed values of P_u or P_d. (So only 3 of the 4 boundary conditions in (2) are relevant.)

This is a two-point, third-order, boundary-value problem. It can be solved by two different methods: shooting, which is relatively easy to program but which cannot normally be extended to problems in higher dimensions; and a direct finite-difference method, which is more complicated to set up but adaptable to more complex situations. The relatively straightforward problem given by (1) and (2) can be used to explore the relative merits of each method; both methods can be used to explore the fluid mechanics of collapsible channel flow.

2 The shooting method

Use a Runge-Kutta routine, or equivalent (e.g. the MATLAB function `ode45`), to integrate (1) from $X = 1$ to $X = 0$ (say), by fixing Q and P_d, setting $H(1) = 1$, $H'(1) = \beta$, $H''(1) = -P_d$ and then varying β until $H(0) = 1$. If you know that a solution exists for $\beta_1 < \beta < \beta_2$, say, a root-finding routine will be useful in quickly homing in to the required solution. Note that you should not use a boundary-value problem solver such as the MATLAB functions `bvp4c` or `bvp5c`, or equivalent.

You should check that your predicted channel shapes and pressure distributions are of sufficient accuracy by varying any tolerance you have specified on the step-length, etc. You will also need to compute solutions with P_u fixed, shooting from $X = 0$ to $X = 1$ (see Question 3 below).

3 The direct finite-difference method

Writing (1) in the form $H_{XXX}H^3 = Q$, discretise this equation and the boundary conditions with second-order accurate finite differences on a uniform N-node grid with grid-spacing $\Delta = 1/(N-1)$ and grid points $X_j = (j-1)\Delta$ $(j = 1, \ldots, N)$. Use forward (or backward) differences for the discretisation of the second derivative in the upstream (or downstream) pressure boundary condition. In most of the interior domain you can use a central difference expression for the discretisation of H_{XXX}, but near one of the boundaries of the domain you will have to use a non-central difference expression; suitable difference formulae are given in Appendix A.

The three discretised boundary conditions and the discretised ODE, written at $(N-3)$ interior gridpoints X_j,$(j = 3, \ldots, N-1)$, provide a total of N non-linear algebraic equations $\mathcal{F}_i(H_j) = 0$ $(i, j = 1, \ldots, N)$ for the discrete membrane heights $H_j = H(X_j)$. Solve this set of equations with a Newton-Raphson method (e.g. [2]).

The Newton-Raphson method requires the Jacobian matrix of the non-linear equations

$$J_{ij} = \partial \mathcal{F}_i / \partial H_j \,,$$

which can be determined by differentiating the discretised equations. At each stage of the iteration, the method requires solution of a set of linear equations. You might find the MATLAB function `spdiags` useful here (see also `help sparfun` in MATLAB).

You should include a brief summary of the equations needed for this method in your write-up.

4 Continuation techniques

The Newton-Raphson method usually requires a “good” initial guess in order to converge to a solution, so to generate solutions corresponding to strongly deformed channels a continuation technique should be used. (The shooting method can also benefit from this approach, but it is not usually necessary.) Start the computation with parameter values corresponding to a known solution (e.g. a slightly deformed channel with $P_d = 0$, $Q \ll 1$) and use the undeformed channel ($H = 1$) as an initial guess. Having found this solution, slowly increment the parameters Q and P_d to construct solutions with the channel highly deformed.

5 Questions

Throughout this project you should comment on the physical interpretation of your computed results as well as their mathematical or numerical features.

Question 1 Compute some static wall shapes when there is no flux through the channel, with the fluid pressure both positive and negative. These shapes can be determined analytically: compare your predictions using both numerical methods above with the exact analytical results.

By considering analytically the case when $|H-1| \ll 1$, or otherwise, show that for $Q > 0$ there are three possible types of solution, depending on the values of P_u or P_d: those with the channel dilated ($H > 1$ in $0 < X < 1$); those with the channel collapsed ($0 < H < 1$ in $0 < X < L$); and those with both dilation and collapse ($H > 1$ for $0 < X < X_b$ and $H < 1$ for $X_b < X < 1$, for some X_b). Show that both numerical methods predict the same channel shapes and pressure distributions for typical values of Q, P_u and P_d. Comment on the qualitative differences between $Q = 0$ and $Q > 0$.

Question 2 Using either method, produce graphs of ΔP as a function of Q for fixed values of the downstream transmural pressure P_d, showing examples with P_d both positive and negative (consider, say, $-3 \leqslant P_d \leqslant 3$, $0 \leqslant Q \leqslant 6$). Show that the slope of the graph of ΔP versus Q falls as Q increases, and show how the channel shape evolves as this happens. Explain this behaviour in *physical* terms.

Question 3 By shooting from $X = 0$ to $X = 1$, produce graphs of ΔP as a function of Q for fixed values of the upstream transmural pressure P_u, both positive and negative, and again describe the evolution of the channel shape. This case requires slightly more care than Question 2, as you may find that the solution is not unique. Show that for each value of P_u there is a maximum possible flow rate through the channel. Obtain the same graphs using the direct finite-difference method; explain any techniques that you may need to introduce in order to obtain converged solutions. Explain the physical mechanism by which the flux may fall as the pressure drop across the channel is increased.

References

[1] Acheson, D. J. *Elementary Fluid Mechanics*, Oxford University Press, 1990.

[2] Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P. *Numerical Recipes* (available in various editions for different languages). Cambridge University Press, 1992.

A Finite difference formulae

Here is a collection of second-order accurate finite difference expressions for the second and third derivatives of the channel height $[H_j = H((j-1)\Delta)]$:

$$H''(X_j) = \frac{H_{j-1} - 2H_j + H_{j+1}}{\Delta^2}$$

$$H''(X_j) = \frac{2H_j - 5H_{j+1} + 4H_{j+2} - H_{j+3}}{\Delta^2}$$

$$H''(X_j) = \frac{2H_j - 5H_{j-1} + 4H_{j-2} - H_{j-3}}{\Delta^2}$$

$$H'''(X_j) = \frac{H_{j+2} - 2H_{j+1} + 2H_{j-1} - H_{j-2}}{2\Delta^3}$$

$$H'''(X_j) = \frac{3H_{j+1} - 10H_j + 12H_{j-1} - 6H_{j-2} + H_{j-3}}{2\Delta^3}$$

