3 Fluid and Solid Mechanics

3.8 Wind-Forced Ocean Currents (10 units)

This project may well be attempted by someone who has attended the two Fluid Dynamics courses in Part IB and Part II.

1 Theory

Western boundary currents develop on the western sides of the ocean basins in response to wind forcing. Poleward flowing western boundary currents like the Gulf Stream and Kuroshio carry vast amounts of heat from the tropics to the mid-latitudes. Here, you will investigate wind-forced ocean circulation and the formation of western boundary currents in an idealized rectangular ocean basin.

A simple depth-independent model of the wind-forced ocean circulation is described by the governing equation for the streamfunction $\psi(x, y, t)$,

$$\zeta_t + J(\psi, \zeta) + v = -\epsilon\zeta + R\tau \tag{1}$$

in $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ with $\psi = 0$ on the boundaries. x and y are Cartesian coordinates representing eastward and northward directions respectively. The vorticity ζ is related to ψ through the Poisson equation

$$\nabla^2\psi = \zeta, \tag{2}$$

and the x and y components of the velocity, respectively u and v, may be written in terms of ψ as

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \qquad (3a, b),$$

$J(\psi, \zeta)$ is the Jacobian with respect to x and y and is an alternative way of writing the advective derivative term $\mathbf{u}.\nabla\zeta$. The $-\epsilon\zeta$ term on the right-hand side of (1) is a simple representation of the effect of bottom friction on the flow. The constant ϵ is a nondimensional frictional damping rate. The term $R\tau(x, y)$, representing the wind forcing, is equal to the curl of the wind stress. It is convenient to take τ to be a prescribed function of x and y and, when investigating the behaviour of the model, to vary the strength of the forcing by varying the constant R.

The steady state form of (1) may be written in the form

$$\mathbf{u}.\nabla(\zeta + y) = R\tau - \epsilon\zeta, \tag{4}$$

implying that in the absence of wind-forcing and bottom friction the quantity $\zeta_a = \zeta + y$ would be conserved following the fluid motion. ζ_a is known as the 'absolute vorticity' and is the vertical (i.e. perpendicular to the Earth's surface) component of the vorticity measured with respect to an inertial frame (i.e. including the Earth's rotation as well as the motion of fluid relative to the Earth). The y contribution to the absolute vorticity is a simple representation of the variation of the vertical component of the rotation vector with latitude.

Question 1 Use incompressibility of $\mathbf{u}$ to rewrite the left hand side of (4). By integrating (4) over a region enclosed by a streamline and, using the divergence theorem on the left-hand side, deduce that if τ is one-signed then no steady state is possible if $\epsilon = 0$, i.e. friction is essential in the steady-state balance.

In this project you will be concerned with steady-state solutions to (1), and their variation as ϵ and R are changed. However, a convenient way to find the steady-state solution is to integrate (1) in time, say from initial conditions in which $\psi = 0$ for all x and y, until the steady state is achieved.

2 Numerical solution of (1)

Define a rectangular grid covering the domain, with points

$$(x_i, y_j) = \left(\frac{i}{N_x}, \frac{j}{N_y}\right) \qquad 0 \leqslant i \leqslant N_x, 0 \leqslant j \leqslant N_y.$$

The grid spacings are $\delta_x = 1/N_x$ and $\delta_y = 1/N_y$ in the x and y directions respectively. The variables ζ and ψ are defined at each point on the grid and it is helpful to use the notation, $\psi_{i,j}^t = \psi(x_i, y_j, t)$, $\zeta_{i,j}^t = \zeta(x_i, y_j, t)$. (The superscripts denote the time at which a particular quantity is to be evaluated.)

In order to integrate (1) in time it is sensible to use the $\zeta_{i,j}^t$ as the working independent variable and derive all the other quantities by solving (2) and then using finite-difference approximations for spatial derivatives in (1). You are recommended to use the expressions

$$v_{i,j}^t \;=\; \frac{\psi_{i+1,j}^t - \psi_{i-1,j}^t}{2\delta_x} \tag{5a}$$

for v and

$$\begin{aligned} J_{i,j}^t = [(&\psi_{i+1,j+1}^t - \psi_{i-1,j+1}^t)\zeta_{i,j+1}^t - (\psi_{i+1,j-1}^t - \psi_{i-1,j-1}^t)\zeta_{i,j-1}^t \\ &- (\psi_{i+1,j+1}^t - \psi_{i+1,j-1}^t)\zeta_{i+1,j}^t + (\psi_{i-1,j+1}^t - \psi_{i-1,j-1}^t)\zeta_{i-1,j}^t]/4\delta_x\delta_y, \end{aligned} \tag{5b}$$

for the Jacobian, both evaluated at the point (x_i, y_j) and time t.

To begin, it is recommended that to integrate (1) in time using the explicit Euler scheme in the form

$$\zeta_{i,j}^{t+\Delta t} - \zeta_{i,j}^t + J_{i,j}^t \Delta t + v_{i,j}^t \Delta t = -\tfrac{1}{2}\epsilon \left(\zeta_{i,j}^{t+\Delta t} + \zeta_{i,j}^t\right) \Delta t + R\tau_{i,j}\Delta t. \tag{6}$$

for $1 \leqslant i \leqslant N_x - 1$, $1 \leqslant j \leqslant N_y - 1$. Note that the boundary condition on ψ means that evaluating $J_{i,j}$ via (5b) at points immediately adjacent to the boundary does not require knowledge of ζ on the boundary itself. There is no need to impose or determine ζ on the boundary at any stage.

> **Question 2** Write a program to integrate the above. Take $\tau = -\sin \pi x \sin \pi y$. You may use a library routine for the solution of Poisson's equation (see Appendix below). Try using a grid size $N_x = N_y = 32$. Note that numerical accuracy of the time integration is not particularly important here because it is only the final steady state that is of interest. You should experiment to find the largest possible time step Δt for which the integration remains stable and approaches a steady state.
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> Concentrate first on the case where R is very small. For $\epsilon = 0.2$ and $\epsilon = 0.05$ produce a plot to verify that your solution approaches a steady state. Plot contour maps of the streamfunction and vorticity fields for the steady state solution. Describe your results in qualitative terms.
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> In this regime you may assume that ζ and ψ scale with R and thus you might find it helpful to redefine ζ and ψ as $\hat{\zeta} = \zeta/R$ and $\hat{\psi} = \psi/R$. Then in the limit $R \to 0$ the

nonlinear term involving the Jacobian may be neglected, and the steady state form of (1) can be approximated as

$$\hat{v} = -\epsilon\hat{\zeta} + \tau\,. \tag{7}$$

You should see that the solution for $\epsilon = 0.05$ is highly asymmetric in the x direction with a strong narrow flow close to the $x = 0$ boundary and a broad weaker flow in the interior. Which term in the equation (1) leads to this asymmetry? For the case $\epsilon = 0.05$, indicate which terms in the equation play a dominant role in the balance in different parts of the flow. Provide an argument for why a strong current near the $x = 1$ boundary cannot exist. In this linear (i.e. small R) case, estimate the maximum value for ψ in the small-ϵ limit. (Consider where a boundary layer may lie, and which boundary conditions you can discard.) Repeat your integrations at higher resolution with $N_x = N_y = 64$. What are your conclusions? What resolution do you think would be needed to adequately resolve the boundary current near $x = 0$?

Question 3 Now, for $\epsilon = 0.05$ investigate the steady-state behaviour as R increases through the range 5×10^{-4} to 10^{-1}. Continue to use $N_x = N_y = 32$. You will find that the timestep Δt must be reduced as R increases, firstly to suppress numerical instability and secondly to allow a steady state to be achieved. For $R = 0.1$ you will need to run your code for a while in order for a steady state to be achieved. You might find it helpful to replace (6) with a more accurate time-stepping scheme such as a 3rd order accurate Runge-Kutta or Adams-Bashforth scheme. This should allow you to take larger time steps, but it is not necessary and full marks can be obtained using (6).

Describe how the pattern of flow changes as R is increased. You may find it useful to look at contour plots of ψ and ζ and also of the quantity $y + \zeta$. Include sufficient plots in your report to illustrate your main points. By considering plots of ψ and ζ and $y + \zeta$, for large R identify the terms involved in the dominant balance of equation (1). Plot a graph of the maximum value of the streamfunction in the domain, $\psi_{\max}$ against R, and try to explain its form. How would you expect $\psi_{\max}$ to depend on the frictional damping rate ϵ in the large-R and small-R limits?

3 Reference

This topic is discussed in some detail in Chapter 19 of the book by Vallis (Vallis, G.K., 2017: *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press; reference copies and an online version are available from the Betty and Gordon Moore Library) but it is certainly not necessary to understand all of this Chapter in detail in order to complete this project.

Appendix

A solver of the Poisson equation in (2) for ψ given $\zeta^t_{i,j}$, is provided on the CATAM website, located among the data files. After copying these matlab files into your working directory, you should be able to call the function poisson which is described below.

[psi]=poisson(x,N,f)

The solver assumes a square grid and that ψ is 0 on all 4 boundaries.

x: vector with grid locations in the x direction (and equivalently in the y direction). The first and last gridpoints in x should correspond to the boundary locations.

N: Type of integration scheme. For 5 point N=5, 9 point N=9, modified 9 point N=10. For more information see reference below.

f: Function to be integrated, in this case ζ. Note that f should be a matrix of size (length(x),length(x)).

psi: The solution to the Poisson equation on the *interior* gridpoints (excluding the values on the boundary which are set to 0). psi will be a matrix of size (length(x)-2,length(x)-2).

For more information on Poisson.m go to `https://cs.nyu.edu/~harper/poisson.htm`. Note that while the mathematics involved in the version described at the link are the same, the inputs of the Poisson solver provided have been modified slightly.

