2 Waves

2.10 Phase and Group Velocity (8 units)

Part II Waves is helpful but not essential. The main prerequisite is an elementary knowledge of the method of stationary phase, as taught in Part II Waves and Part II Asymptotic Methods. However, all the required material can be found in [1], or any of the books listed in the Asymptotic Methods schedule, or those by Billingham & King, Lighthill and Whitham in the Waves schedule.

The Klein-Gordon equation,

$$u(x,t): \quad u_{tt} - c_0^2 u_{xx} = -q^2 u \quad \text{with } c_0 > 0 \text{ and } q \geqslant 0 \text{ constants} , \tag{1}$$

arises, for example, when looking for solutions of 'the' wave equation $p_{tt} - c_0^2 \nabla^2 p = 0^*$ in the form $p = u(x,t)\cos(m\pi y/a)\cos(n\pi z/b)$ to describe the propagation of, say, sound waves in a rectangular tube ('waveguide') occupying $0 < y < a$, $0 < z < b$. It is a simple example of a wave equation which is both *hyperbolic* (there is an upper bound on the speed at which waves can propagate) and *dispersive* (different Fourier components propagate at different speeds).

1 Initial-value problem

The goal of this part of the project is to solve (1) for $x \in (-\infty, \infty)$ with initial conditions

$$u(x,0) = f(x) , \quad u_t(x,0) = 0 , \tag{2}$$

where $f(x)$ is some specified (real) function with $f(x) \to 0$ as $|x| \to \infty$, and which for simplicity we shall assume to be even in x, in which case the solution $u(x,t)$ will be even in x for all t. Without loss of generality we set $c_0 = 1$. (Why?)

When $q = 0$, (1) reduces to 'the' one-dimensional wave equation familiar from IB Methods, and the solution subject to (2) is

$$u(x,t) = \tfrac{1}{2}\left[f(x-t) + f(x+t)\right] . \tag{3}$$

Question 1 Show that for general q, (1)–(2) may be solved by a Fourier transform in x to give

$$u(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx - i\Omega(k)t} dk + \text{complex conjugate} \tag{4}$$

where

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{5}$$

and

$$\Omega(k) = \sqrt{q^2 + k^2} \quad \text{(the 'dispersion relation').} \tag{6}$$

Note that if $f(x)$ is even in x, $\tilde{f}(k)$ is real (and even in k).

Sketch graphs† of 'phase velocity' $\Omega(k)/k$, and 'group velocity' $\Omega'(k)$ against k (on the same axes). Give a physical interpretation of phase velocity.

*More properly, this should be called 'the linear non-dispersive wave equation', for there are many other equations which describe wave propagation, the Klein-Gordon equation being one such.

† Whereas almost all graphs, including labels, annotations, etc., need to be computer-generated, this is one of the relatively few cases where a scanned hand-drawing **is** acceptable for electronic submission.

An alternative representation of the solution, which you are *not* asked to verify‡, is

$$u(x,t) = \tfrac{1}{2}\left[f(x-t) + f(x+t) - qt\int_{-\pi/2}^{\pi/2} J_1\left(qt\cos\theta\right) f\left(x + t\sin\theta\right) \mathrm{d}\theta\right] \tag{7}$$

where $J_1(x)$ is the Bessel function of the first kind [which could be computed by using the MATLAB function besselj(1,x)]. Neither analytic form is particularly suitable for calculating $u(x,t)$ at a large number of points. Nevertheless, the Fourier representation is convenient in allowing a simple approximation for large t to be obtained by the *method of stationary phase*.

Question 2 Giving only a brief outline of the theory, show by the method of stationary phase applied to the integral (4) that for $q > 0$, in the limit $t \to \infty$ with $V \equiv x/t$ fixed and $|V| < 1$

$$u(x,t) \sim \left[2\pi\Omega''(k_0)t\right]^{-1/2} |\tilde{f}(k_0)| \cos\left[k_0 x - \Omega(k_0)t + \arg\tilde{f}(k_0) - \tfrac{1}{4}\pi\right] \tag{8}$$

where k_0 is specified by

$$\Omega'(k_0) = V \ , \tag{9}$$

and hence (assuming that $\tilde{f}(k)$ is real)

$$u(x,t) \sim \frac{q^{1/2}t}{(2\pi)^{1/2}\left(t^2 - x^2\right)^{3/4}}\, \tilde{f}\left(\frac{qx}{\sqrt{t^2 - x^2}}\right) \cos\left(q\sqrt{t^2 - x^2} + \tfrac{1}{4}\pi\right) \quad \text{for } |x| < t \ . \tag{10}$$

Explain how this provides a physical interpretation of group velocity.

Question 3 Write a program to solve (1)–(2) numerically as follows: take a uniform grid with steplengths Δt in t and Δx in x, and use the centred-difference approximation

$$u_{tt}(x,t) = \frac{u(x, t+\Delta t) - 2u(x,t) + u(x, t-\Delta t)}{(\Delta t)^2} + O\left((\Delta t)^2\right) \ , \tag{11}$$

and a similar one for u_{xx}, to generate the finite-difference scheme

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} - \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} = -q^2\left(\frac{u_{i+1}^j + u_{i-1}^j}{2}\right) \tag{12}$$

where u_i^j is an approximation to $u\left(i\Delta x, j\Delta t\right)$. (It is possible to use the more natural $-q^2 u_i^j$ on the right-hand side, but then a smaller Δt may be required for numerical stability: see [2], p.588.) The solution at t-level $j+1$ can thus be found from those at j and $j-1$. Clearly the first t-level needs special treatment: for the initial conditions (2) it is sufficient, to the order of accuracy of the scheme, to set $u_i^{-1} = u_i^1$. (Why?) Further, since we are restricting attention to solutions even in x, we need only solve in $0 \leqslant x \leqslant L \equiv N\Delta x$ with the additional boundary condition $u_x(0,t) = 0$, which can be implemented in the finite-difference scheme by setting $u_{-1}^j = u_1^j$. Note that the u_0^j are determined as part of the solution, but the outer-boundary values u_N^j need to be specified.

‡The derivation is indicated in [3]

Run the program with

$$f(x) = \begin{cases} (1-x^2)^2 & |x| \leqslant 1 \\ 0 & |x| \geqslant 1 \end{cases} , \tag{13}$$

for both $q = 0$ and $q = 1$, at least up to $t = 50$. The x-range should be sufficiently large that the outer-boundary values u_N^j can safely be set equal to zero. Experiment with different values of Δx and $\Delta t/\Delta x$, and comment on the accuracy and stability of the numerical scheme; you may wish to compare with the exact solutions (3) and/or (7) at selected points.

Find $\tilde{f}(k)$ and plot its graph.

Plot the finite-difference solutions against x at various representative values of t, using steplengths which are sufficiently small for graphical accuracy; for the case $q = 1$, superpose the stationary-phase approximation (10). Comment on the solutions, mentioning similarities and differences between those for $q = 0$ and $q = 1$. What would you expect to happen if q were non-zero but very small?

2 Signalling Problem

Question 4 Modify your program to solve (1) for $x > 0$, $t > 0$ with the initial conditions

$$u(x, 0) = u_t(x, 0) = 0$$

and the boundary condition

$$u(0, t) = \sin(\omega_0 t) \ .$$

Physically, this describes a system which is undisturbed ($u \equiv 0$) for $t < 0$, but excited for $t > 0$ by a time-harmonic forcing applied at $x = 0$. As before, there is no loss of generality in taking $c_0 = 1$. (Why?)

For $q = 0$, find the solution analytically, and use it as a check on the program.

Run the program with $q = 1$ and $\omega_0 = 0.9$, 1.1 and 1.5, at least as far as $t = 150$, with suitable steplengths and domain size (give brief justification for your choice). For each case, present *selected* results to illustrate the key features. Comment on the solutions, particularly in the light of phase and group velocities (see Question 1).

References

[1] Carrier, G.F., Crook, M. & Pearson, C.E., *Functions of a Complex Variable.*

[2] Dodd, R.K., Eilbeck, J.C., Gibbon, J.D., & Morris, H.C., *Solitons and Nonlinear Wave Equations.*

[3] Ockendon, J.R., Howison, S.R., Lacey, A.A, & Movchan, A.B., *Applied Partial Differential Equations.*

