23 Astrophysics

23.6 Accretion Discs (8 units)

No knowledge of astrophysics is assumed or required in this project. All relevant equations are defined and explained in the project.

1 Fluid Equations

Accretion discs are composed of fluid orbiting a central object. They evolve viscously so that matter falls inwards while angular momentum drifts outwards. Accretion discs are found in binary star systems, around forming stars and in active galactic nuclei. We use cylindrical polar coordinates (R, ϕ, z) and assume axisymmetry for this project. When the central object dominates the gravitational field the angular velocity of the matter in the disc is Keplerian so that

$$\Omega = \left(\frac{GM}{R^3}\right)^{1/2}, \tag{1}$$

where M is the mass of the central object and G is the gravitational constant. The disc is made up of annuli of matter lying between R and $R + \Delta R$ with mass $2\pi R \Delta R \Sigma$, where $\Sigma(R, t)$ is the surface density (with dimensions ML^{-2}) of the disc at time t.

The equation describing conservation of mass is

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma V_{\mathrm{R}}) = 0, \tag{2}$$

where V_{R} is the radial velocity in the disc. Conservation of angular momentum gives us

$$R\frac{\partial(\Sigma R^2 \Omega)}{\partial t} + \frac{\partial}{\partial R}(R\Sigma V_{\mathrm{R}} R^2 \Omega) = \frac{1}{2\pi}\frac{\partial \Gamma}{\partial R}, \tag{3}$$

where the viscous torque

$$\Gamma = 2\pi R \nu \Sigma R^2 \Omega' \tag{4}$$

where $\Omega' = \frac{d\Omega}{dR}$ and $\nu(R, \Sigma)$ is the viscosity in the disc.

Question 1

Show that, for Ω independent of time,

$$R\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial R}\left[\frac{1}{2\pi (R^2\Omega)'}\frac{\partial \Gamma}{\partial R}\right]. \tag{5}$$

and hence, using equation 1, for a Keplerian disc that

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(3\nu\Sigma R^{1/2})\right] \tag{6}$$

This is the basic equation that governs the evolution of the surface density of a Keplerian accretion disc, which we shall assume for the rest of this project. The viscosity may be a function of Σ, R and t and so this equation may be non-linear.

Question 2

The mass accretion rate $\dot{m}$ through the disc is

$$\dot{m}(R) = -2\pi R\Sigma V_{\rm R} \tag{7}$$

Show that

$$V_{\rm R} = -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2}). \tag{8}$$

In a steady state disc the accretion rate through the disc is constant. Find, analytically, the steady state solution for $\nu\Sigma$ from equation (6). Use the inner boundary condition $\Sigma = 0$ at $R = R_{\rm in}$ where ν is finite at $R = R_{\rm in}$. Plot $\nu\Sigma$ in units of $\dot{m}$ against $R/R_{\rm in}$ in the range $1 < R/R_{\rm in} < 100$.

We can make the problem dimensionless by setting $r = R/R_0$, $\tau = t/t_0$, $\sigma(r,\tau) = \Sigma/\Sigma_0$, and $\eta(r,\sigma) = \nu/\nu_0$. Find a condition for t_0 such that equation (6) remains the same for these dimensionless variables. Assume that the accretion rate is small enough that $M \approx$ const.

Question 3

Use the substitution $X = r^{1/2}$ in equation 6 to derive

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 g}{\partial X^2} \tag{9}$$

where f and g are to be determined. We consider X to be discrete with values X_i where $i = 1, 2, ...100$ and $X_1 = \Delta X$ where ΔX is a constant. We let $X_{i+1} = X_i + \Delta X$ and time be τ_n where $n = 1, 2, ...$ with $\tau_{n+1} = \tau_n + \Delta\tau$. We have $\tau_1 = 0$ and we let $f_i^n = f(X_i, \tau_n)$ and $g_i^n = g(X_i, \tau_n)$.

Show that equation (9) can be represented by the difference equation

$$f_i^{n+1} = f_i^n + \frac{\Delta\tau}{(\Delta X)^2}(g_{i+1}^n - 2g_i^n + g_{i-1}^n). \tag{10}$$

Question 4

Write a program to solve equation (9) taking $\eta = 1$ and with the boundary conditions $\sigma(r_{\rm in}, \tau) = 0$ and $\sigma(r_{\rm out}, \tau) = 0$. Set $r_{\rm in} = 0.0004$ and $r_{\rm out} = 4$ and use 100 grid points equally spaced in X between $X = 0.02$ and $X = 2$. For formal stability the timestep must satisfy

$$\Delta\tau \leqslant \frac{1}{2}(\Delta X)^2\frac{f}{g} \tag{11}$$

at all points in the disc. However you may find that you need to use something smaller. Evolve from an initial mass distribution

$$\sigma(r, 0) = \exp\left(-\frac{(r^{1/2}-1)^2}{0.001}\right). \tag{12}$$

Plot the initial surface density, $\sigma(r, 0)$, against r. Plot σ against r at times $\tau = 0.002$, 0.008, 0.032, 0.128 and 0.512 on the same axes.

Question 5

Adapt your program so that you can find the height and position of the peak in the surface density. Make a table of the surface density at the peak and the position of the peak at the times used in question 4. Furthermore, adapt your program to plot the time evolution of the total angular momentum in the disc (normalised to its value at $t = 0$), and the position of the peak angular momentum surface density ($R^2\Omega\Sigma$) as a function of time.

Comment on the difference in behaviour between the surface density and angular momentum.

Question 6

The evolution of a particle in the disk is given by $dR/dt = V_\mathrm{R}(R, t)$ where V_R is given by equation 8. Rewrite equation 8 in a form similar to equation 10 so that you can adapt the code written for question 4 to plot radial velocity (in dimensionless units) as a function of radius for the timesteps used in question 4 on the same figure, taking care with choice of axis to show where this is positive and negative.

Question 7

Use the radial velocities found by your code to follow the evolution of particles' orbits, and plot that evolution up to $\tau = 1$ for particles initially at $r_0 = 0.9, 0.95, 1.0, 1.05, 1.1$. Plot the maximum radius attained by a particle as well as the time it took to reach that distance and, for those particles that do so, the time it takes to reach a boundary as a function of its initial radius for the range $r_0 = 0.9 - 1.1$.

Question 8

Use the results from question 7 to work out the range of initial radii r_0 that have reached the inner boundary by $\tau = 0.512$ and hence, using equation 12, the fraction of the initial mass that has reached the inner boundary by this time. Also do the corresponding calculation for the outer boundary. Compare these values with the fraction of the initial mass that remains at that time that you derive using the surface density profile from question 4.

MATLAB Specific Issues

Use of the MATLAB in-build function GRADIENT is prohibited, as its algorithm or accuracy is not documented.

