23 Astrophysics

23.4 Stellar Structure (8 units)

This project is concerned with the structure of stars. All relevant equations are defined and explained in the project itself, and hence no course in Part II of the Mathematical Tripos is a pre-requisite. For those students taking Part II of the Astrophysics Tripos, knowledge of the 'Structure and Evolution of Stars' course is useful.

1 Introduction

The physics of stars can be encapsulated in a set of differential equations which can be solved with appropriate boundary conditions. The most efficient method of solving the equations is by a relaxation technique. In order to converge, it relies on an initial guess to the solution that is not far from the actual solution. In this project we construct such an initial model of a star by a shooting technique that directly integrates the equations from the boundaries, where conditions are varied, until the two solutions meet in the middle.

2 The Equations of Stellar Structure

The structure of a spherically symmetric star of uniform and unchanging composition, in thermal equilibrium, can be described by four non-linear differential equations in five variables together with an equation of state and boundary conditions.

1) Hydrostatic Equilibrium,

$$\frac{dp}{dr} = -\rho \frac{\mathrm{G}m}{r^2}, \tag{1}$$

where p is the pressure and ρ the density at radius r, measured from the centre, m is the mass interior to r and G is Newton's gravitational constant $(6.6726 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}})$.

2) Mass continuity,

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{2}$$

3) Energy generation,

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon, \tag{3}$$

where L_r is the luminosity, the outward flow of energy through a sphere at radius r, and

$$\epsilon = \epsilon(\rho, T, \text{composition}) \tag{4}$$

is the energy generation rate per unit mass.

4) Energy can be transported by radiation (or equivalently conduction) or by bulk convective motions. In the radiative case

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi \mathrm{a c} r^2 T^3}, \tag{5}$$

where

$$\kappa = \kappa(\rho, T, \text{composition}) \tag{6}$$

is the opacity, a is the radiation constant $(7.5646 \times 10^{-16}\mathrm{J\,m^{-3}K^{-4}})$ and c the speed of light in a vacuum $(2.9979 \times 10^8\,\mathrm{m\,s^{-1}})$.

The equation of state relates pressure to density, temperature and composition throughout the star,

$$p = p(\rho, T, \text{composition}). \tag{7}$$

Appropriate boundary conditions at the centre are

$$m = 0, \quad L_r = 0 \quad \text{at} \quad r = 0. \tag{8}$$

At the surface, $r = R_*$, the radius of the photosphere, an Eddington approximation to a plane parallel grey* atmosphere leads to

$$L_* = 4\pi R_*^2 \sigma T^4, \tag{9}$$

where $\sigma = \mathrm{ac}/4$ is the Stefan–Boltzmann constant, L_* is the bolometric luminosity of the star and

$$p\kappa = \frac{2}{3}\frac{\mathrm{G}M_*}{R_*^2}, \tag{10}$$

where M_* is the stellar mass. There are four independent variables $p(r)$, $m(r)$, L_r and $T(r)$, when $\rho(r)$ is determined by equation (7), for which a unique solution can be found.

2.1 Choice of variables

In practice it is better to use m rather than r as independent variable and, for uniform composition, the solution is unique for a given stellar mass. We then apply the surface boundary conditions at $m = M_*$.

> **Question 1** Find the derivatives with respect to m of the new dependent variables, r^3, L_r, T^4 and $\ln p$.

2.2 Equations at the centre

As they stand the equations are not suitable for numerical integration at the centre. It is therefore necessary to develop them to obtain conditions at some small but finite value of r.

> **Question 2** Develop the differential equations obtained in Question 1 at the centre to obtain equations such as
>
> $$p = p_\mathrm{c} - \frac{2}{3}\pi \mathrm{G}\rho_c^2 r^2 \tag{11}$$
>
> for small r, where p_c and ρ_c are the central values. Do not forget to take account of the central boundary conditions.

3 The Physics of the Equation of State, Energy Generation and Opacity

For the purposes of this project we shall assume that stars are composed entirely of hydrogen (mass fraction X, assumed to be 0.7 throughout this project) and helium (mass fraction $Y = 1 - X$) and that the contributions to pressure, other than that of the perfect gas, are negligible so that

$$p = \frac{\rho \mathrm{R}^* T}{\mu}, \tag{12}$$

*A grey atmosphere is one in which the opacity is independent of wavelength.

where R* is the gas constant ($8.3145 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$) and μ is the mean molecular weight. It is sufficient to calculate μ on the assumption that the material is completely ionized so that

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y. \tag{13}$$

The same applies to γ (see question 4), the adiabatic exponent, which may be taken to be constant at $\frac{5}{3}$ for a monatomic ideal gas. We shall further assume that the opacity, κ, may be approximated by the contribution from electron scattering

$$\kappa_{\mathrm{es}} = 0.02(1+X)\mathrm{m^2\,kg^{-1}}. \tag{14}$$

Finally we assume that nuclear burning proceeds via a combination of the proton-proton chain and CNO cycle, producing energy at a rate

$$\epsilon = \left(0.25\,X^2 \mathrm{e}^{-33.8T_6^{-1/3}} + 8.8\times10^{18} X \mathrm{e}^{-152.28T_6^{-1/3}}\right) T_6^{-2/3} \frac{\rho}{\mathrm{kg\,m^{-3}}} \mathrm{W\,kg^{-1}} \tag{15}$$

where $T_6 = T/10^6\,\mathrm{K}$.

Question 3 The Sun is observed to have mass $M_\odot = 1.9891\times10^{30}\,\mathrm{kg}$, radius $R_\odot = 6.9598\times10^8\,\mathrm{m}$ and luminosity $L_\odot = 3.8515\times10^{26}\,\mathrm{W}$. Estimate the temperature at which nuclear reactions can halt the gravitational collapse of a solar mass star and argue that this is a good initial guess for T_c. Also, make a linear approximation to the pressure gradient through the Sun, choosing a sensible boundary condition for the pressure at the the stellar surface, to obtain an estimate of the central pressure p_c.

4 A Shooting Solution

We are now set up to find a numerical solution to the equations. A logical way to proceed is to guess central values of the variables, T_c and p_c and integrate the equations to the surface where the solution will not necessarily fit the boundary conditions. Unfortunately, it turns out that such direct integrations either from the centre to the surface or from the surface to the centre diverge unacceptably at the surface or the centre for small changes in the undetermined boundary variables (the equations are non-linear). They are, however, well behaved in between. We can therefore integrate both outwards from the centre and inwards from the surface (guessing the radius $r = R_*$ and luminosity $L_r = L_*$ at $m = M_*$) and meet at some point in the interior $m = M_\mathrm{s}$. We can then vary R_*, L_*, p_c and T_c until the two solutions are continuous at $m = M_\mathrm{s}$. *Hint: The boundary conditions need to be specified at the centre ($M_r = 0$) and surface ($M_r = M_*$), then quantities found from solving the model ODEs.*

The questions below seek solutions for a star of mass $3\,M_\odot$.

Question 4 Use the shooting method to solve for a simplified stellar structure from equations (1) and (2). Just for this question, make the approximation that $T \propto \rho^{\gamma-1}$, and take the radius of the star to be $R_* = 1.5\,R_\odot$. Integrate both inwards and outwards to some suitable intermediate point (e.g., $M_\mathrm{s} = 0.5M$). Describe how the boundary conditions at the surface and centre are implemented in your integrations. Using the differences of the dependent variables at M_s as a measure of the quality of the fit, discuss how the fit varies with p_c and T_c and use this knowledge to find values for p_c and T_c that match the dependent variables at M_s to within 1% of their values there. Provide a plot with your best solution. Also indicate on this plot how the solution changes with p_c and T_c.

Question 5 Now include equations (3), (5) and the associated boundary conditions to implement the shooting method to find a more detailed stellar structure model. Describe how the new boundary conditions differ from those used in question 4. From question 4 you have a reasonable first estimate for p_c, T_c and R_*, but an estimate of L_* is required because a $3\,M_\odot$ star is significantly brighter than the Sun. An estimate may be made by shooting inwards from the surface to close to the centre using the complete set of equations. Discuss, and illustrate concisely, how the values of each of the dependent variables close to the centre vary as L_* is varied. Use this knowledge to give an estimate of L_*. *Hint: When do the estimates of L_* become unphysical?*

Question 6 Now shoot both in and out to an intermediate point, and as before, use the differences of the dependent variables at that point to determine the quality of the solution. Use your initial estimates for R_*, L_*, p_c and T_c, and your observations of how the fit changes as these parameters are varied, to describe how to obtain by hand a relative difference in the dependent variables at M_s that is less than 10%. Provide your best-fitting parameters, the relative difference in each of the dependent variables at M_s and a plot of each dependent variable against m.

Adjusting the solution by hand is not a very effective way of refining the parameters to get an accurate solution. Let $\Delta x_i(R_*, L_*, p_\mathrm{c}, T_\mathrm{c}), i \in 1, 2, 3, 4$ be the differences between the inward and outward values of each variable x_i at the intermediate point. A better estimate for these parameters can be found by applying the correction $(\Delta R_*, \Delta L_*, \Delta p_\mathrm{c}, \Delta T_\mathrm{c})$ from the solution to the matrix equation:

$$\begin{pmatrix} \frac{\partial \Delta x_1}{\partial R_*} & \frac{\partial \Delta x_1}{\partial L_*} & \frac{\partial \Delta x_1}{\partial p_\mathrm{c}} & \frac{\partial \Delta x_1}{\partial T_\mathrm{c}} \\ \frac{\partial \Delta x_2}{\partial R_*} & \frac{\partial \Delta x_2}{\partial L_*} & \frac{\partial \Delta x_2}{\partial p_\mathrm{c}} & \frac{\partial \Delta x_2}{\partial T_\mathrm{c}} \\ \frac{\partial \Delta x_3}{\partial R_*} & \frac{\partial \Delta x_3}{\partial L_*} & \frac{\partial \Delta x_3}{\partial p_\mathrm{c}} & \frac{\partial \Delta x_3}{\partial T_\mathrm{c}} \\ \frac{\partial \Delta x_4}{\partial R_*} & \frac{\partial \Delta x_4}{\partial L_*} & \frac{\partial \Delta x_4}{\partial p_\mathrm{c}} & \frac{\partial \Delta x_4}{\partial T_\mathrm{c}} \end{pmatrix} \begin{pmatrix} \Delta R_* \\ \Delta L_* \\ \Delta p_\mathrm{c} \\ \Delta T_\mathrm{c} \end{pmatrix} = - \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{pmatrix}_{(R_*, L_*, p_\mathrm{c}, T_\mathrm{c})}, \tag{16}$$

where $\frac{\partial \Delta x_i}{\partial R_*}$ can be determined by computing $\Delta x_i(R_* + dR_*, L_*, p_\mathrm{c}, T_\mathrm{c})$, and likewise for the other partial derivatives. This procedure can be iterated until a desired accuracy is achieved.

Question 7 Implement the matrix method described above and iterate until the parameters are accurate to within four significant figures. Provide plots of each variable with respect to m, a table with the values of each variable at the meeting point and your values of the parameters. *Hint: You may need to rescale the variables and parameters so that the matrix is well-conditioned and the inversion is stable.*

Question 8 The model you find does not agree with observations of such stars. Suggest what needs to be added to the model to make it more realistic.

References

[1] Phillips, A.C. *The Physics of Stars*, Wiley, 1999.

[2] Prialnik, D. *An Introduction to the Theory of Stellar Structure and Evolution* (Second Edition), Cambridge University Press, 2009.

