23 Astrophysics

23.1 Generating a Consistent Self-Gravitating System (8 units)

The project is self-contained, though some knowledge of galactic structure may be advantageous.

Introduction: distribution functions of gravitating systems

A collection of N particles moving under their mutual gravitational attraction only, can be well described by a continuous *distribution function*, $f(\mathbf{x}, \mathbf{v}, t)\, d^3\mathbf{x}\, d^3\mathbf{v}$, in the limits where N is large and the particles have some positions $\mathbf{x}_i$ and velocities $\mathbf{v}_i$, $i \in \{1, \ldots, N\}$, at time t.

In the limit where two–body interactions can be neglected (N large enough), the flow in phase space is said to be *collisionless* and the distribution function satisfies the Boltzmann equation:

$$\frac{df}{dt} = 0. \tag{1}$$

The dynamics of the system are governed by the Poisson equation,

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x}), \tag{2}$$

where the spatial density is $\rho(\mathbf{x}, t) = \int f d^3\mathbf{v}$, Φ is the potential and G is the gravitational constant.

Instructions

The aim of this project is to generate a discrete particle realisation of a spherical self-gravitating system, given a potential–density pair describing the system, and compare the results for a finite number of particles with those from continuous distributions. Here we will consider a specific example of a spherical, isotropic distribution, the so-called γ distribution functions ([2]). The γ functions have density profiles that are proportional to $r^{-\gamma}$ as $r \to 0$. Here we consider $\gamma = 0$.

For a spherical, isotropic system, f can be expressed as a function of the specific energy, E, only. For a particle i, $E_i = \frac{1}{2}v_i^2 + \Phi(r_i)$, where $r_i = |\mathbf{x}_i|$ and $v_i = |\mathbf{v}_i|$.

Question 1 The $\gamma = 0$ model is specified by the density profile

$$\rho(r) = \frac{3Ma}{4\pi(r+a)^4}. \tag{3}$$

The associated potential is determined by solving (2). Show that in this case

$$\Phi(r) = -\frac{GM}{2a}\left[\frac{(r+a)^2 - r^2}{(r+a)^2}\right] \tag{4}$$

where M is the total mass.

The enclosed mass within radius r is given by

$$M(r) = \int_0^r 4\pi\rho(r')r'^2 dr' = M\left(\frac{r}{r+a}\right)^3.$$

Without loss of generality, we take $a = 1$, $G = 1$, and $M(r \to \infty) = 1$. Additionally define the dimensionless binding energy $\epsilon = -E \times (a/GM)$ and potential $\Psi = -\Phi \times (a/GM)$.

Given $\rho(\Phi)$, it is straightforward to derive $f(\epsilon)$ with an Abel transform ([1], especially page 651). For this model

$$f(\epsilon) = \frac{3}{4\sqrt{2}\pi^3}\int_0^\epsilon \frac{(1-y)^2(2y+4y^2)}{y^4\sqrt{\epsilon-\Psi}}d\Psi, \tag{5}$$

where $y(\Psi) = \sqrt{1-2\Psi}$.

Question 2 Verify that

$$f(\epsilon) = \frac{3}{2\pi^3}\left[\frac{(3-4\epsilon)\sqrt{2\epsilon}}{1-2\epsilon} - 3\sinh^{-1}\sqrt{\frac{2\epsilon}{1-2\epsilon}}\right]. \tag{6}$$

The incremental mass of particles, dM, with binding energies in the energy interval ϵ to $\epsilon + d\epsilon$ is given by the differential energy distribution

$$\frac{dM}{d\epsilon} = f(\epsilon)g(\epsilon), \tag{7}$$

where the density of states $g(\epsilon)$ ([2]) is given by

$$g(\epsilon) = 8\pi^2\left[\sqrt{1-2\epsilon}\,\frac{3-14\epsilon-8\epsilon^2}{12\epsilon^2} - \pi + \frac{1-6\epsilon+16\epsilon^2}{(2\epsilon)^{5/2}}\cos^{-1}(-\sqrt{1-2\epsilon})\right]. \tag{8}$$

We wish to generate a realisation of our ($\gamma = 0$) distribution, using a Monte–Carlo acceptance–rejection algorithm, as discussed below.

Programming Task: Generate an $N = 5000$ particle realisation of the distribution function above. You should truncate the distribution at some finite radius r_T (recommended values are 100 or 300) and renormalise your particle mass after generating the realisation so that $M = 1$.

To do this you may find the following information useful: the maximum value of the quantity $r^2v^2 \times f(\epsilon)$ is 2.884×10^{-3}; the maximum binding energy $\epsilon_{\max}$ is $\frac{1}{2}$; and the maximum speed of a particle $v_{\max}$ is $\sqrt{2\epsilon_{\max}}$.

One way to approach this problem is to draw a pair of uniform random numbers with $r_r \in [0, r_T)$ and $v_r \in [0, v_{\max})$. Then compare the quantity $(r_r/a)^2(v_r^2/(M/a))f(\epsilon(r_r, v_r))$ and its maximum value with another uniform random variable ξ_i and accept or reject your draw from phase space accordingly.

Comparisons with analytic results

Question 3 Compare your numerical energy distribution $dM(\epsilon)/d\epsilon$ with the expected analytic differential energy distribution.

Question 4 Given your set of N particles, r_i, v_i, $i = 1, \ldots, N$, generate a uniform three-dimensional realisation of your distribution in Cartesian co-ordinates. That is, form the set $\{x_i, y_i, z_i, v_{xi}, v_{y_i}, v_{zi}\}$ by Monte-Carlo generation of a uniform distribution of Cartesian components of r_i, v_i.

What is the actual mass of your N particles and how does it compare with the mass you expected given the choice of r_T? You may want to try different values of r_T to see how $M(r_T)$ varies with r_T.

Question 5 Write a short routine that sorts the particles into radius bins and generates a numerical density profile of your distribution. How does your actual density profile compare with the expected analytic density profile? How does the density profile fit change as you vary your bin size? (**Hint**: use a log–log plot.)

Question 6 Show that for this distribution the dispersion $\sigma^2(r) = \langle v^2 \rangle$, where angle brackets denote the average value over the particles, is given by

$$\sigma^2(r) = \frac{GM(a + 6r)}{10(r + a)^2}. \qquad (9)$$

Compare your numerical dispersion with the analytic estimate. Calculate the angular momentum $\mathbf{L}(r, \Delta r) = \sum_{r<r_i<r+\Delta r} m_i \mathbf{x}_i \times \mathbf{v}_i$, in radial bins. Does your distribution have any net angular momentum? Should it?

Question 7 The anisotropy is defined as $\beta(r) = 1 - \langle v_t^2 \rangle / 2 \langle v_r^2 \rangle$, where $\langle v_t^2 \rangle = \langle v^2 \rangle - \langle v_r^2 \rangle$ and $\langle v_r^2 \rangle = \langle (\mathbf{v} \cdot \mathbf{x}/r)^2 \rangle$. Plot the anisotropy as a function of radius. Is your distribution anisotropic? Should it be?

Question 8 As a function of radius, what is the potential of your realisation and how does it compare with the analytic potential estimated? How does the $\langle v^2 \rangle$ compare with the local escape speed as a function of radius? How does varying the particle number affect your results?

References

[1] Binney, J. and Tremaine, S. *Galactic Dynamics*, Princeton Series in Astrophysics, 1987.

[2] Dehnen, W. *A family of potential–density pairs for spherical galaxies and bulges*, *MNRAS*, **265**, 250–256 (1993). Full article available online from NASA ADS, `http://ukads.nottingham.ac.uk/abstract_service.html`

