20 Probability

20.6 Loss Networks (9 units)

This project requires knowledge of discrete- and continuous-time Markov chains, covered in the Part IB Markov Chains and Part II Applied Probability courses respectively.

Introduction

Nearly a century ago the mathematician Erlang, working for the Copenhagen Telephone Company, devised the first mathematical theories for telecommunications networks. The technology we have now would seem like science fiction to Erlang, yet his insight into the essential structure of networks means that his theorems are just as useful for designing an optically-routed backbone for the Internet as they were for early Danish telephony.

In this project we will deal with certain models for telephone networks, arising from Erlang's work. We consider a network to be a collection of links (cables). Each telephone call occupies a certain amount of space on certain links; for example, a telephone call from a Cambridge college to a London house might occupy 8 kbit/sec of space on the link from the college to the university exchange, and on the link from the university exchange to BT's Cambridge exchange, and on the link from there to a London exchange, and so on. This space is occupied for the duration of the call. The links which comprise the national telephone network only have limited capacity, and when they are full we get a busy signal. Some interesting questions are: What is the probability of a busy signal? How does it depend on the volume of traffic? Can we reduce this probability by strategies such as offering multiple routes for a call?

For further reading see [1].

1 The Erlang link

Consider a single link with the capacity to carry C simultaneous calls. Suppose that new calls arrive as a Poisson process of rate ν, that each call lasts for a duration which is exponential with mean 1, and that all call durations are independent of each other and of the arrival process. If a new call arrives when the link is already carrying C calls, then the new call is *blocked*. This system is known as the "Erlang link".

> **Question 1** Set up a continuous-time Markov model for the Erlang link. Calculate the equilibrium probability that there are i calls in progress.

Define $E(\nu, C)$ to be the equilibrium probability that there are C calls in progress.

We say that an arriving call "sees" the system in state s if the system is in state s just before it arrives. The PASTA property (Poisson arrivals see time averages) says that the long-run proportion of arrivals which see the system in state s is equal to the equilibrium probability that the system is in state s.

> **Question 2** Show that the PASTA property holds for the Erlang link. *Hint. One approach is to consider the discrete-time Markov chain which records the state of the system after each event, where an event is an attempted arrival or a departure, and to*

express the equilibrium distribution of this chain in terms of the equilibrium distribution for the continuous-time chain. Deduce that the long-run proportion of calls which are blocked is $E(\nu, C)$.

Question 3 Write a program to simulate the Erlang link and to measure the blocking probability. Compare the empirical blocking probability to that given by $E(\nu, C)$ for a range of values of C up to 600 and an appropriate range of values of ν.

2 Alternative Routing

Consider now a network of links. For simplicity, suppose that the network is a complete graph on K nodes, i.e., there is a link between every pair of nodes $\{1, \ldots, K\}$, and that each link has capacity C. Suppose that for every pair of nodes (a, b), calls between a and b arise as a Poisson process of rate ν. It might seem reasonable, in order to reduce the blocking probability, to offer an alternative route if the direct link is full. Specifically, suppose that calls between a and b are routed as follows:

1. If there is spare capacity on the direct link $a \leftrightarrow b$, route the call over that link.
2. Otherwise, pick a new node c uniformly at random from the other $K - 2$ nodes. If there is spare capacity on $a \leftrightarrow c$ and on $c \leftrightarrow b$, route the call over these two links.
3. Otherwise, the call is blocked.

We will call this the "Alternative Routing" system. One way (but not the only way) to obtain a 'merit' mark in this project is to prove that the Alternative Routing system satisfies the PASTA property.

As you can see, it is possible in principle to set up a Markov process model for this system, and thereby to calculate the equilibrium distribution; but the number of states is so large that, even for moderate K, it is not computationally practical to do so. Instead, we can use a famous approximation called the *Erlang fixed point approximation*, which is that blocking occurs independently on different links. This leads to the formula

$$B = E(\nu + 2\nu B(1 - B), C) \tag{1}$$

where B is the probability that an incoming call cannot be routed on its chosen direct link.

Question 4 Give a careful intuitive explanation for (1). In what sense could $\nu + 2\nu B(1 - B)$ be called the "offered load" on a link?

Question 5 Demonstrate numerically (e.g., by plotting an appropriate graph) that for some values of C and ν this equation has a unique solution, and that for other values it has multiple solutions.

Now pick $C = 600$ and choose ν such that (1) has multiple solutions.

Question 6 Write a program to simulate the Alternative Routing system. Explain clearly and concisely the algorithm you have used. Run your simulation, for $K = 5$, and record the cumulative count of the number of calls blocked. Plot a graph which shows this count as a function of time. *Programming hint. Run your simulation for at least a million transitions of the Markov process; this should take less than 10 minutes on a modern computer. You may find it convenient to write a small subsample of your simulation output to a file, then use Excel or some other program to plot the result.*

You should find, from your simulation, that the system spends some of the time in a high-blocking regime and some of the time in a low-blocking regime, corresponding to solutions of (1).

Question 7 Give an intuitive explanation for why there are multiple solutions. It is a standard result from Markov chain theory that this Markov process has a unique equilibrium distribution; comment briefly on how this result relates to the existence of multiple solutions.

Question 8 You should observe that there is one solution to (1) which is *not* reflected in your simulations. By considering fixed points of the map $B \leftarrow E(\nu + 2\nu B(1-B), C)$, suggest why this is so.

Question 9 Use the Erlang fixed-point approximation to find the probabilities that an incoming call is blocked in the high-blocking regime and in the low-blocking regime. How do these compare to a network without alternative routing, i.e., in a network in which a call may only be routed on the direct link?

3 Trunk reservation

A telecoms operator would be alarmed at the situation you investigated in the last section, and would seek to control the network so that it stays in the low-blocking state. One way to do this is with a technique called "trunk reservation". We modify the call admission procedure, so that calls arising between a and b are routed as follows:

1. Consider the direct link $a \leftrightarrow b$. If this link has spare capacity, route the call over it.
2. Otherwise, pick a new node c uniformly at random from the other $K-2$ nodes, and consider the two links $a \leftrightarrow c$ and $c \leftrightarrow b$. If on each of these links the number of calls in progress is less than $C-s$, then route the call over these two links.
3. Otherwise, the call is blocked.

The parameter $s > 0$ is known as the *trunk reservation parameter*.

Question 10 Develop a fixed-point approximation for this system. *Hint. First set up a suitable Markov model for the number of calls in progress on a single link with two classes of traffic.*

Question 11 Compare the blocking probability to those you found in Question 9. Has trunk reservation improved matters? How large should the trunk reservation parameter be?

References

[1] F. P. Kelly, *Network Routing*, Philosophical Transactions of the Royal Society series A, 1991. `http://www.statslab.cam.ac.uk/~frank/loss/`

